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Improved approximation algorithms for weighted 2-path partitions☆,☆☆

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ABSTRACT

We investigate two NP-complete vertex partition problems on edge-weighted complete graphs with $3k$ vertices. The first problem asks to partition the graph into k vertex disjoint paths of length 2 (referred to as 2-*paths*) such that the total weight of the paths is maximized. We present a cubic time approximation algorithm that computes a 2-path partition whose total weight is at least .5833 of the weight of an optimal partition, improving upon the $(.5265 - \epsilon)$-approximation algorithm of Tanahashi and Chen (2010). Restricting the input to graphs with edge weights in $\{0, 1\}$, we present a .75 approximation algorithm improving upon the .55-approximation algorithm of Hassin and Schneider (2013).

Combining this algorithm with a previously known approximation algorithm for the 3-Set Packing problem, we obtain a .6-approximation algorithm for the problem of partitioning a $\{0, 1\}$-edge-weighted graph into k vertex disjoint triangles of maximum total weight. The best known approximation algorithm for general weights is due to Chen and Tanahashi (2009) and achieves an approximation ratio of .5257.

1. Introduction

For some integer $k > 0$ let G be a complete graph on $3k$ vertices having non-negative edge weights. The Maximum Weight 2-path Partition (M2PP) problem asks to compute a set of k vertex disjoint paths of length 2 (referred to as 2-paths) such that the sum of the weights of the paths is maximized. The Maximum Weight Triangle Partition (MTP) problem asks to compute a set of k vertex disjoint cycles of length 3 (referred to as triangles) such that the sum of the edge weights of the k cycles is maximized. For the remainder of this work G will refer to a complete graph on $3k$ vertices having non-negative edge weights.

Our main contribution is a $7/12$-approximation algorithm for the M2PP problem. We also investigate $\{0, 1\}$-edge-weighted graphs – in which the weights of the edges of the input graph are either 0 or 1 – and we present a .75-approximation algorithm for the M2PP problem and a .6-approximation algorithm for the MTP problem in this setting.

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Both the M2PP and MTP problems have been studied before, mostly under the names Maximum 2-Path Packing and Maximum Triangle Packing respectively. Unfortunately, these names have also been used for the related but different problems defined below. Consequently, we use separate terminology to make a clear distinction between the packing and partitioning settings.

Given an unweighted graph H, a 2-Path Packing of H is a collection of vertex disjoint 2-paths and a Triangle Packing of H is a collection of vertex disjoint triangles. Such a collection is called *perfect* if it uses all the vertices of H. The Maximum 2-Path Packing problem asks to find a 2-Path Packing of maximum cardinality. The Maximum Triangle Packing problem is defined similarly.

1.1. Related work

In their classic book Garey and Johnson [9] show that deciding whether a graph admits a perfect Triangle Packing or a perfect 2-Path Packing is NP-complete (p. 68 and 76 respectively). More general results on the NP-completeness of packing families of graphs into a given graph are shown by Hell and Kirkpatrick [16], and Lonc [21]. Both the Maximum 2-Path Packing and Maximum Triangle Packing problems are special cases of the unweighted 3-Set Packing problem for which Hurkens and Schrijver [17] (also see Halldórsson [11]) presents a local search algorithm that achieves a $\frac{2}{3}-\epsilon$ approximation (with $\epsilon > 0$).

Kann [18] shows that the Maximum Triangle Packing is APX-hard even in graphs of maximum degree 4 and Chlebík and Chlebíková [6] show that Maximum Triangle Packing is NP-hard to approximate within a factor of .9929. Moreover, Guruswami et al. [10] show that the problem remains NP-complete even when restricted to the families of chordal, planar, line or total graphs. A .833-approximation algorithm for graphs with maximum degree 4 is presented by Manić and Wakabayashi [22].

For Maximum 2-Path Packing problem van Bevern et al. [27] study its computational complexity on multiple classes of special graphs. The authors provide a quasilinear-time algorithm for the problem on interval graphs and polynomial time algorithms for the more general Star Partition problem on cographs and on bipartite permutation graphs. Moreover, the authors show that the problem is *NP*-hard even on grid graphs with maximum degree three. Prieto and Sloper [24] present a fixed parameter tractable algorithm and Babenko and Gusakov [2] present an approximation algorithm for an edge-weighted version of the problem.

The M2PP and MTP problems studied in this paper are special cases of the weighted 3-Set Packing problem for which Arkin and Hassin [1] present a $\frac{1}{2}-\epsilon$ approximation algorithm. M2PP and MTP are special cases because all sets of size three exist and have non-negative weights, and the weight of each set is determined by the weights of the included subsets of size two. Hassin et al. [13] observe that there exists a simple reduction from M2PP (respectively, MTP) to the problem of deciding whether a graph has a perfect 2-Path Packing (resp., Triangle Packing), implying that the two problems are NP-complete. Moreover, they present a randomized $\frac{35}{67}-\epsilon \approx .5222$ approximation algorithm for M2PP and Tanahashi and Chen [26] refine and derandomize this algorithm, leading to an improved approximation ratio of $.5265-\epsilon$; van Zuylen [28] presents a simpler analysis of this algorithm. For the MTP problem, Hassin and Shlomi Rubinstein [13] (see also Erratum [14]) presents a $\frac{43}{83}-\epsilon \approx .518$ approximation algorithm and Chen et al. [4] (see also Erratum [5]) present a randomized approximation algorithm which achieves a ratio of .5257. For $\{0, 1\}$-edge-weighted graphs, Hassin and Schneider [15] present a local search based .55-approximation algorithm for M2PP which runs in time $O(|V|^{10})$. Hassin and Rubinstein [12] also study the problem of partitioning a complete weighted graph into paths of length 3, and present a .75-approximation algorithm.

1.2. Contribution

In this paper we present a simple, matching based $7/12 \approx .583$-approximation algorithm for the M2PP problem on graphs with general non-negative weights, improving upon the $(.5265-\epsilon)$-approximation algorithm of Tanahashi and Chen [26]. Besides improving the approximation ratio, our algorithm is significantly less computationally intensive since the constant factor for the algorithm of Tanahashi and Chen is exponential in $1/\epsilon$. Moreover, for $\{0, 1\}$-edge-weighted graphs we provide a .75-approximation algorithm for the M2PP problem improving upon the .55-approximation algorithm of Hassin and Schneider [15]. The core idea of our algorithms is adapted from Hassin and Rubinstein [12], where the authors show how to partition a complete weighted graph into paths of length 3. For a complete graph on $n = 3k$ vertices we prove the following two theorems in Section 3 and Section 4 respectively.

Theorem 1. *There exists a $7/12$-approximation algorithm for the* M2PP *problem running in time $O(n^{2.5})$.*

Theorem 2. *For $\{0, 1\}$-edge-weighted graphs there exists a $3/4$-approximation algorithm for the* M2PP *problem running in time $O(n^{2.5})$.*

In Section 5 we show how an approximation algorithm for M2PP can be combined with a 3-Set Packing approximation algorithm to obtain an approximate solution for the MTP on $\{0, 1\}$-edge-weighted graphs. We are not aware of any previous results for the MTP problem restricted to this case.

Theorem 3. *For $\{0, 1\}$-edge-weighted graphs there exists a $5/8$-approximation algorithm for the* MTP *problem running in time $O(n^{2.5})$.*