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Degenerate matchings and edge colorings

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ABSTRACT

A matching M in a graph G is r-degenerate if the subgraph of G induced by the set of vertices incident with an edge in M is r-degenerate. Goddard, Hedetniemi, Hedetniemi, and Laskar (Generalized subgraph-restricted matchings in graphs, Discrete Mathematics 293 (2005) 129–138) introduced the notion of acyclic matchings, which coincide with 1-degenerate matchings. Solving a problem they posed, we describe an efficient algorithm to determine the maximum size of an r-degenerate matching in a given chordal graph. Furthermore, we study the r-chromatic index of a graph defined as the minimum number of r-degenerate matchings into which its edge set can be partitioned, obtaining upper bounds and discussing extremal graphs.

1. Introduction

Matchings in graphs are a central topic of graph theory and combinatorial optimization [25]. While classical matchings are tractable, several well known types of more restricted matchings, such as induced matchings [8,31] or uniquely restricted matchings [17], lead to hard problems. Goddard, Hedetniemi, Hedetniemi, and Laskar [15] proposed to study so-called subgraph-restricted matchings in general. In particular, they introduce the notion of acyclic matchings. By a simple yet elegant argument (cf. Theorem 4 in [15]), they show that finding a maximum acyclic matching in a given graph is hard in general, and they explicitly pose the problem to describe a fast algorithm for the acyclic matching number in interval graphs. In the present paper, we solve this problem for the more general chordal graphs. Furthermore, we study the edge coloring notion corresponding to acyclic matchings.

Before we give exact definitions and discuss our results as well as related research, we introduce some terminology. We consider finite, simple, and undirected graphs, and use standard notation. A *matching* in a graph G is a subset M of the edge set $E(G)$ of G such that no two edges in M are adjacent. Let $V(M)$ be the set of vertices incident with an edge in M. A matching M is *induced* [8] if the subgraph $G[V(M)]$ of G induced by the set $V(M)$ is 1-regular, that is, M is the edge set of $G[V(M)]$. Induced matching are also known as *strong* matchings. A matching M is *uniquely restricted* [17] if there is no other matching M' in G distinct from M that satisfies $V(M) = V(M')$. It is easy to see that M is uniquely restricted if and only if there is no M-alternating cycle in G, which is a cycle in G every second edge of which belongs to M [17]. Finally, M is *acyclic* [15] if $G[V(M)]$ is a forest. Let $\nu(G)$, $\nu_s(G)$, $\nu_{ur}(G)$, and $\nu_1(G)$ be the maximum sizes of a matching, an induced matching, a uniquely restricted matching, and an acyclic matching in G, respectively. Since every induced matching is acyclic, and every acyclic matching is uniquely restricted, we have

$$\nu_s(G) \leq \nu_1(G) \leq \nu_{ur}(G) \leq \nu(G).$$

We chose the notation "$\nu_1(G)$" rather than something like "$\nu_{ac}(G)$", because we consider a further natural generalization.

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For a non-negative integer r, a graph G is *r-degenerate* if every subgraph H of G of order at least 1 has a vertex of degree at most r in H. Note that a graph is a forest if and only if it is 1-degenerate. An *r-degenerate order* of a graph G is a linear order $u_1, \dots, u_n$ of its vertices such that, for every i in $[n]$, the vertex u_i has degree at most r in $G[\{u_i, \dots, u_n\}]$, where $[n]$ is the set of the positive integers at most n. Clearly, a graph is r-degenerate if and only if it has an r-degenerate order.

Now, let a matching M in a graph G be *r-degenerate* if the induced subgraph $G[V(M)]$ is r-degenerate, and let $\nu_r(G)$ denote the maximum size of an r-degenerate matching in G.

For every type of matching, there is a corresponding edge coloring notion. An *edge coloring* of a graph G is a partition of its edge set into matchings. An edge coloring is *induced (strong)*, *uniquely restricted*, and *r-degenerate* if each matching in the partition has this property, respectively. Let $\chi'(G)$, $\chi'_s(G)$, $\chi'_{ur}(G)$, and $\chi'_r(G)$ be the minimum numbers of colors needed for the corresponding colorings, respectively. Clearly,

$$\chi'_s(G) \geq \chi'_1(G) \geq \chi'_{ur}(G) \geq \chi'(G).$$

In view of the hardness of the restricted matching notions, lower bounds on the matching numbers [18–21], upper bounds on the chromatic indices [3,4], efficient algorithms for restricted graph classes [9–11,13,26], and approximation algorithms have been studied [4,30]. There is only few research concerning acyclic matchings; Panda and Pradhan [29] describe efficient algorithms for chain graphs and bipartite permutation graphs.

Vizing's [32] famous theorem says that the *chromatic index* $\chi'(G)$ of G is either $\Delta(G)$ or $\Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. Induced edge colorings have attracted much attention because of the conjecture $\chi'_s(G) \leq \frac{5}{4}\Delta(G)^2$ posed by Erdős and Nešetřil (cf. [12]). Building on earlier work of Molloy and Reed [28], Bruhn and Joos [7] showed $\chi'_s(G) \leq 1.93\Delta(G)^2$ provided that $\Delta(G)$ is sufficiently large. In [4] it is shown that $\chi'_{ur}(G) \leq \Delta(G)^2$ with equality if and only if G is the complete bipartite graph $K_{\Delta(G),\Delta(G)}$.

Our results are upper bounds on $\chi'_r(G)$ with the discussion of extremal graphs, and an efficient algorithm for $\nu_r(G)$ in chordal graphs, solving the problem posed in [15].

2. Bounds on the r-degenerate chromatic index

Since, for every two positive integers r and Δ, every r-degenerate matching of the complete bipartite graph $K_{\Delta,\Delta}$ of order 2Δ has size at most r, we obtain $\chi'_r(K_{\Delta,\Delta}) \geq \frac{\Delta^2}{r}$.

Our first result gives an upper bound in terms of r and Δ.

Theorem 1. *If r is a positive integer and G is a graph of maximum degree at most Δ, then*

$$\chi'_r(G) \leq \frac{2(\Delta-1)^2}{r+1} + 2(\Delta-1) + 1. \tag{1}$$

Proof. Let $K = \left\lfloor \frac{2(\Delta-1)^2}{r+1} + 2(\Delta-1) + 1 \right\rfloor$. The proof is based on an inductive coloring argument. We may assume that all but exactly one edge uv of G are colored using colors in $[K]$ such that, for every color α in $[K]$, the edges of G colored with α form an r-degenerate matching. We consider the colors in $[K]$ that are forbidden by colors of the edges close to uv. In order to complete the proof, we need to argue that there is always still some available color for uv in $[K]$.

Recall that $N_G(u)$ is the neighborhood $\{v \in V(G) : uv \in E(G)\}$ of u in G, and that $N_G[u]$ is the closed neighborhood $\{u\} \cup N_G(u)$ of u in G.

We introduce some notation illustrated in Fig. 1. Let $N_u = N_G(u) \setminus N_G[v]$, $N_v = N_G(v) \setminus N_G[u]$, and $N_{u,v} = N_G(u) \cap N_G(v)$. Let $n_u = |N_u|$, $n_v = |N_v|$, and $n_{u,v} = |N_{u,v}|$. Clearly, $n_u + n_{u,v} = d_G(u) - 1 \leq \Delta - 1$ and $n_v + n_{u,v} = d_G(v) - 1 \leq \Delta - 1$. Let E_u be the set of edges between u and N_u, E_v be the set of edges between v and N_v, $E_{u,v}$ be the set of edges between $\{u, v\}$ and $N_{u,v}$, and, for every vertex $w \in N_u \cup N_v \cup N_{u,v}$, let E_w be the set of edges incident with w but not incident with u or v. Clearly, $|E_u| + |E_v| + |E_{u,v}| = (d_G(u)-1) + (d_G(v)-1) \leq 2(\Delta-1)$ and $|E_w| \leq \Delta - 1$ for every vertex $w \in N_u \cup N_v \cup N_{u,v}$.

Let $\mathcal{F}_1$ be the colors that appear on edges in $E_u \cup E_v \cup E_{u,v}$. Clearly, every color in $\mathcal{F}_1$ is forbidden for uv, because each color class must be a matching. Let $\mathcal{F}_2$ be the colors α in $[K]$ that do not belong to $\mathcal{F}_1$ such that

$$d_u^\alpha + 2d_{u,v}^\alpha + d_v^\alpha \geq r+1,$$

where d_u^α is the number of vertices in N_u incident with an edge colored α, d_v^α is the number of vertices in N_v incident with an edge colored α, and $d_{u,v}^\alpha$ is the number of vertices in $N_{u,v}$ incident with an edge colored α. Note that, since $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, none of the edges contributing to $d_u^\alpha + 2d_{u,v}^\alpha + d_v^\alpha$ is incident with u or v.

If there is some α in $[K] \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$, then neither u nor v is incident with an edge of color α, and $d_u^\alpha + 2d_{u,v}^\alpha + d_v^\alpha \leq r$. This implies $\min\{d_u^\alpha + d_{u,v}^\alpha, d_v^\alpha + d_{u,v}^\alpha\} \leq \lfloor r/2 \rfloor \leq r-1$ and $\max\{d_u^\alpha + d_{u,v}^\alpha, d_v^\alpha + d_{u,v}^\alpha\} \leq r$. Hence, coloring uv with color α, the vertices incident with the edges of G colored α induce an r-degenerate graph G_α. In fact, if $d_u^\alpha + d_{u,v}^\alpha \leq d_v^\alpha + d_{u,v}^\alpha$, then u has $d_u^\alpha + d_{u,v}^\alpha + 1 \leq r$ neighbors in G_α, and v has $d_u^\alpha + d_{u,v}^\alpha + 1 \leq r+1$ neighbors in G_α, one of which is u. Since $G_\alpha - \{u, v\}$ is r-degenerate by assumption, it follows that G_α is r-degenerate.

As explained above this would complete the proof. Therefore, we may assume that $\mathcal{F}_1 \cup \mathcal{F}_2 = [K]$.

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