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On integer images of max-plus linear mappings

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ABSTRACT

Let us extend the pair of operations $(\oplus, \otimes) = (\max, +)$ over real numbers to matrices in the same way as in conventional linear algebra.

We study integer images of mappings $x \to A \otimes x$, where $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$. The question whether $A \otimes x$ is an integer vector for at least one $x \in \mathbb{R}^n$ has been studied for some time but polynomial solution methods seem to exist only in special cases. In the terminology of combinatorial matrix theory this question reads: is it possible to add constants to the columns of a given matrix so that all row maxima are integer? This problem has been motivated by attempts to solve a class of job-scheduling problems.

We present two polynomially solvable special cases aiming to move closer to a polynomial solution method in the general case.

1. Introduction

Since the 1960s max-algebra provides modelling and solution tools for a class of problems in discrete mathematics and matrix algebra. The key feature is the development of an analogue of linear algebra for the pair of operations $(\oplus, \otimes)$ where

$$a \oplus b = \max(a, b)$$

and

$$a \otimes b = a + b$$

for $a, b \in \overline{\mathbb{R}} \stackrel{def}{=} \mathbb{R} \cup \{-\infty\}$. This pair is extended to matrices and vectors as in conventional linear algebra. That is if $A = (a_{ij}),\ B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if

$$c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$$

for all i, j. If $\alpha \in \overline{\mathbb{R}}$ then $\alpha \otimes A = \left(\alpha \otimes a_{ij}\right)$. For simplicity we will use the convention of not writing the symbol $\otimes$. Thus in what follows the symbol $\otimes$ will not be used (except when necessary for clarity), and unless explicitly stated otherwise, all multiplications indicated are in max-algebra.

The interest in max-algebra (today also called tropical linear algebra) was originally motivated by the possibility of dealing with a class of non-linear problems in pure and applied mathematics, operational research, science and engineering as if they were linear due to the fact that $\left(\overline{\mathbb{R}}, \oplus, \otimes\right)$ is a commutative and idempotent semifield. Besides the main advantage of using linear rather than non-linear techniques, max-algebra enables us to efficiently describe and deal with complex sets [6], reveal combinatorial aspects of problems [5] and view a class of problems in a new, unconventional way. The first

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pioneering papers appeared in the 1960s [17,18] and [36], followed by substantial contributions in the 1970s and 1980s such as [19,23,24,37] and [16]. Since 1995 we have seen a remarkable expansion of this research field following a number of findings and applications in areas as diverse as algebraic geometry [31] and [35], geometry [27], control theory and optimization [1], phylogenetic [34], modelling of the cellular protein production [3] and railway scheduling [25]. A number of research monographs have been published [1,7,25] and [30]. A chapter on max-algebra appears in a handbook of linear algebra [26] and a chapter on idempotent semirings can be found in a monograph on semirings [22].

Max-algebra covers a range of linear-algebraic problems in the max-linear setting, such as systems of linear equations and inequalities, linear independence and rank, bases and dimension, polynomials, characteristic polynomials, matrix scaling, matrix equations, matrix orbits and periodicity of matrix powers [1,7,19,14] and [25]. Among the most intensively studied questions was the *eigenproblem*, that is the question, for a given square matrix A to find all values of λ and non-trivial vectors x such that $Ax = \lambda x$. This and related questions such as z-matrix equations $Ax \oplus b = \lambda x$ [15] have been answered [10,19,24,20,2] and [7] with numerically stable low-order polynomial algorithms. The same applies to the *subeigenproblem* that is the problem of finding solutions to $Ax \leq \lambda x$ [33] and the *supereigenproblem* that is solution to $Ax \geq \lambda x$, [8] and [32]. Max-linear and integer max-linear programs have also been studied [37,7,9,21] and [13].

A specific area of interest is in solving the above mentioned problems with integrality requirements. It seems in general there is no polynomial solution method to find an integer eigenvector of a real matrix in max-algebra or to decide that there is none. A closely related [13] is the question whether the mapping $x \to Ax$ has an integer image, that is whether Ax is an integer vector for at least one $x \in \mathbb{R}^n$. The motivation for the latter comes from operational problems such as the following job-scheduling task [19] and [7]: Products $P_1, \ldots, P_m$ are prepared using n machines (processors), every machine contributing to the completion of each product by producing a component. It is assumed that each machine can work for all products simultaneously and that all these actions on a machine start as soon as the machine starts to work. Let a_{ij} be the duration of the work of the jth machine needed to complete the component for P_i ($i = 1, \ldots, m; j = 1, \ldots, n$). If this interaction is not required for some i and j then a_{ij} is set to $-\infty$. The matrix $A = \left(a_{ij}\right)$ is called the *production matrix*. Let us denote by x_j the starting time of the jth machine ($j = 1, \ldots, n$). Then all components for P_i ($i = 1, \ldots, m$) will be ready at time

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}).$$

Hence if $b_1, \ldots, b_m$ are given completion times then the starting times have to satisfy the system of equations:

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = b_i \text{ for all } i = 1, \ldots, m.$$

Using max-algebra this system can be written in a compact form as a system of linear equations:

$$Ax = b. \tag{1}$$

A system of the form (1) is called a *one-sided system of max-linear equations* (or briefly a *one-sided max-linear system* or just a *max-linear system*). Such systems are easily solvable [17,37] and [7], see also Section 2. However, sometimes the vector b of completion times is not given explicitly, instead it is only required that completions of individual products occur at discrete time intervals, for instance at integer times. This motivates the study of integer images of max-linear mappings to which this paper aims to contribute. More precisely, we deal with the question: Given a real matrix A, find a real vector x such that Ax is integer or decide that none exists. In the terminology of combinatorial matrix theory this question reads: is it possible to add constants to the columns of a given matrix so that all row maxima are integer? We will call this problem the *Integer Image Problem* (IIP). This problem has been studied for some time [12,13] and [29], yet it seems to be still open whether it can be answered in polynomial time. In this paper we present two polynomially solvable special cases aiming to suggest a direction in which an efficient method could be found for general matrices in the future. We also provide a brief summary of a selection of already achieved results.

2. Definitions, notation and previous results

Throughout the paper we denote $-\infty$ by ε (the neutral element with respect to $\oplus$) and for convenience we also denote by the same symbol any vector, whose all components are $-\infty$, or a matrix whose all entries are $-\infty$. A matrix or vector with all entries equal to 0 will also be denoted by 0. If $a \in \mathbb{R}$ then the symbol a^{-1} stands for $-a$. Matrices and vectors whose all entries are real numbers are called *finite*. We assume everywhere that $m, n \geq 1$ are integers and denote $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$.

It is easily proved that if A, B, C and D are matrices of compatible sizes (including vectors considered as $m \times 1$ matrices) then the usual laws of associativity and distributivity hold and also isotonicity is satisfied:

$$A \geq B \Longrightarrow AC \geq BC \text{ and } DA \geq DB. \tag{2}$$