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The strong metric dimension of the power graph of a finite group

Xuanlong Ma [a], Min Feng [b,*], Kaishun Wang [c]

[a] School of Science, Xi'an Shiyou University, Xi'an 710065, China
[b] School of Science, Nanjing University of Science and Technology, Nanjing 210094, China
[c] Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China

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ABSTRACT

We characterize the strong metric dimension of the power graph of a finite group. As applications, we compute the strong metric dimension of the power graph of a cyclic group, an abelian group, a dihedral group and a generalized quaternion group.

1. Introduction

Given a graph Γ, denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set of Γ, respectively. For $x, y, z \in V(\Gamma)$, we say that z *strongly resolves* x and y if there exists a shortest path from z to x containing y, or a shortest path from z to y containing x. A subset S of $V(\Gamma)$ is a *strong resolving set* of Γ if every pair of vertices of Γ is strongly resolved by some vertex of S. The smallest cardinality of a strong resolving set of Γ is called the *strong metric dimension* of Γ and is denoted by $\mathrm{sdim}(\Gamma)$.

In the 1970s, the metric dimension was first introduced, by Harary and Melter [11] and, independently, by Slater [30]. This parameter has appeared in a number of publications (see [3] and [4] for more information). In 2004, Sebő and Tannier [29] introduced the strong metric dimension of a graph and presented some applications of strong resolving sets to combinatorial searching. The strong metric dimension of corona product graphs, rooted product graphs and strong products of graphs were studied in [22–24], respectively. The problem of computing strong metric dimension is NP-hard [23]. Some theoretical results, computational approaches and recent results on strong metric dimension can be found in [20].

On the other hand, graphs associated to various algebraic structures have been actively investigated, since they have valuable applications (cf. [8,19]), are related to automata theory (cf. [13,14]) and make it possible to apply the algebraic tools for helping to solve problems in graph theory and vice versa.

The *power graph* Γ_G of a finite group G has the vertex set G and two distinct elements are adjacent if one is a power of the other. In 2000, Kelarev and Quinn [15] introduced the concept of a power graph. Recently, many interesting results on power graphs have been obtained, see [1,5–7,9,10,16–18,25–27]. A detailed list of results and open questions on power graphs can be found in [2].

This paper is organized as follows. In Section 2, we express the strong metric dimension of a graph with diameter two in terms of the clique number of its reduced graph. Sections 3 and 4 study the clique number of the reduced graph of the power

* Corresponding author.
E-mail addresses: xuanlma@xsyu.edu.cn (X. Ma), fengmin@njust.edu.cn (M. Feng), wangks@bnu.edu.cn (K. Wang).

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graph of a finite group G. Therefore, the strong metric dimension of Γ_G is characterized. In Section 5, we compute the strong metric dimension of the power graph of a cyclic group, an abelian group, a dihedral group and a generalized quaternion group.

2. Reduced graphs

Let Γ be a connected graph. The *distance* $d_\Gamma(x, y)$ between vertices x and y is the length of a shortest path from x to y in Γ. The *closed neighborhood* of x in Γ, denoted by $N_\Gamma[x]$, is the set of vertices which have distance at most one from x. The greatest distance between any two vertices in Γ is called the *diameter* of Γ. A subset of $V(\Gamma)$ is a *clique* if any two distinct vertices in this subset are adjacent in Γ. The *clique number* $\omega(\Gamma)$ is the maximum cardinality of a clique in Γ.

Proposition 2.1. *Let Γ be a connected graph with diameter two. Then a subset S of $V(\Gamma)$ is a strong resolving set of Γ if and only if the following conditions hold:*
(i) *$V(\Gamma) \setminus S$ is a clique in Γ;*
(ii) *$N_\Gamma[u] \neq N_\Gamma[v]$ for any two distinct vertices u and v of $V(\Gamma) \setminus S$.*

Proof. Assume that S is a strong resolving set of Γ. Let u and v be two distinct vertices of $V(\Gamma) \setminus S$. Since Γ has diameter two, we have $d_\Gamma(u, v) = 1$ by [21, Property 2]. This means that (i) holds. If $N_\Gamma[u] = N_\Gamma[v]$, then $d_\Gamma(u, w) = d_\Gamma(v, w)$ for any $w \in S$, and so u and v cannot be strongly resolved by any vertex in S, a contradiction. Hence (ii) holds.

For the converse, it follows from (ii) that there exists a vertex w in $N_\Gamma[u] \setminus N_\Gamma[v]$ or $N_\Gamma[v] \setminus N_\Gamma[u]$. Without loss of generality, let $w \in N_\Gamma[u] \setminus N_\Gamma[v]$. By (i), we have $w \in S$. Note that (w, u, v) is a shortest path. Therefore, w strongly resolves u and v, as desired. □

For vertices x and y in a graph Γ, we write $x \equiv y$ if $N_\Gamma[x] = N_\Gamma[y]$. Observe that $\equiv$ is an equivalence relation. Let $U(\Gamma)$ be a complete set of distinct representative elements for this equivalence relation. The *reduced graph* $\mathcal{R}_\Gamma$ of Γ has the vertex set $U(\Gamma)$ and two vertices are adjacent if they are adjacent in Γ. For two distinct equivalence classes O_1 and O_2, if there exist a vertex in O_1 and a vertex in O_2 which are adjacent in Γ, then each vertex in O_1 and each vertex in O_2 are adjacent in Γ. Hence, the reduced graph $\mathcal{R}_\Gamma$ does not depend on the choice of representatives. We get the following result immediately from Proposition 2.1.

Theorem 2.2. *Let Γ be a connected graph with diameter two. Then*

$$\mathrm{sdim}(\Gamma) = |V(\Gamma)| - \omega(\mathcal{R}_\Gamma).$$

3. The clique number of $\mathcal{R}_G$

In the remainder of this paper, we always use G to denote a finite group. To simplify, denote by $\mathcal{R}_G$ the reduced graph $\mathcal{R}_{\Gamma_G}$. Since the group G is finite, it is obvious that the diameter of Γ_G is at most two. In order to compute $\mathrm{sdim}(\Gamma_G)$, we only need to study $\omega(\mathcal{R}_G)$ from Theorem 2.2.

For a positive integer n, write

$$n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}, \tag{1}$$

where $p_1, p_2, \ldots, p_m$ are pairwise distinct prime numbers and $r_i \geq 1$ for $1 \leq i \leq m$. Write

$$\sigma_n = \begin{cases} 1, & \text{if } m = 1; \\ \sum_{i=1}^{m} r_i, & \text{if } m \geq 2. \end{cases}$$

Let $\mathbb{Z}_n$ be the cyclic group of order n. The following result is our first main theorem, which will be proved in the next section.

Theorem 3.1. $\omega(\mathcal{R}_{\mathbb{Z}_n}) = \sigma_n$.

In the rest of this section, assume that G is a noncyclic group. Denote by $\mathcal{M}$ the set of all maximal cyclic subgroups of G. Given a prime p, let $\mathcal{M}_p$ be the set of all p-subgroups in $\mathcal{M}$. Suppose $\mathcal{M}_p \neq \emptyset$. Let

$$\mathcal{M}_p = \{M_1, M_2, \ldots, M_t\}. \tag{2}$$

For $i \in \{1, \ldots, t\}$, write

$$\{M_i \cap M_j : j \in \{1, \ldots, t\}\} = \{C_{i1}, \ldots, C_{is_i}\}.$$

Note that $C_{i1}, \ldots, C_{is_i}$ are subgroups of M_i which is a cyclic group of prime-power order. Without loss of generality, we may assume that

$$C_{i1} \subsetneq C_{i2} \subsetneq \cdots \subsetneq C_{is_i} = M_i. \tag{3}$$

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