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Universal solvability of interval max-plus matrix equations

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ABSTRACT

This paper deals with the solvability of interval matrix equations in max-plus algebra. Max-plus algebra is the algebraic structure in which classical addition and multiplication are replaced by $\oplus$ and $\otimes$, where $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$.

The notation $\boldsymbol{A} \otimes X \otimes \boldsymbol{C} = \boldsymbol{B}$, where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are given interval matrices, represents an interval max-plus matrix equation. We define three types of solvability of interval max-plus matrix equations, namely the strong universal , universal , and weak universal solvability. We derive the necessary and sufficient conditions which can be verified in polynomial times.

1. Introduction

Behaviour of discrete event systems, in which the individual components move from event to event rather than varying continuously through time, is often described by systems of linear equations or by matrix equations. Discrete dynamic systems and related algebraic structures were studied using max-plus matrix operations in [2,3,16]. In the last decades, significant effort has been developed to study systems of max-plus linear equations in the form $A \otimes x = b$, where A is a matrix, b and x are vectors of compatible dimensions. Systems of linear equations over max-plus algebra are used in several branches of applied mathematics. Among interesting real-life applications let us mention e.g. a large scale model of Dutch railway network or synchronizing traffic lights in Delfts [13]. In the last two decades, interval systems of the form $\boldsymbol{A} \otimes x = \boldsymbol{b}$ have been studied, for details see [2,4–6,9,8,12].

In this paper, we shall deal with interval matrix equations of the form $\boldsymbol{A} \otimes X \otimes \boldsymbol{C} = \boldsymbol{B}$, where A, B, and C are given interval matrices of suitable sizes a X is an unknown matrix. In the following example we show one of the possible applications.

Example 1.1. Consider the manufacturing company that carries its three types of products in three places P_1, P_2, and P_3. These products are placed in two warehouses W_1 and W_2. Afterwards, they are loaded into two trucks T_1 and T_2 which deliver products to three shops S_1, S_2 and S_3 (see Fig. 1).

In Fig. 1, there is an arrow from P_i to W_j if the products made in place P_i are stored in warehouse W_j. If products from warehouse W_j are loaded on a truck T_l, then there is an arrow from W_j to T_l. And, there is an arrow from T_l to S_k, if the truck T_l expedites products to shop S_k ($i = 1, 2, 3; j = 1, 2; k = 1, 2, 3; l = 1, 2$).

The symbols above or below the arrows, a_{ij} (c_{lk}), express known times needed to transport goods from place P_i to warehouse W_j (from truck T_l to shop S_k). The time needed to transport goods from warehouse W_j to truck T_k via W_j and T_l is denoted by x_{jk}. Then, total time needed to transport goods from place P_i to shop S_k is $a_{ij} + x_{jl} + c_{lk}$.

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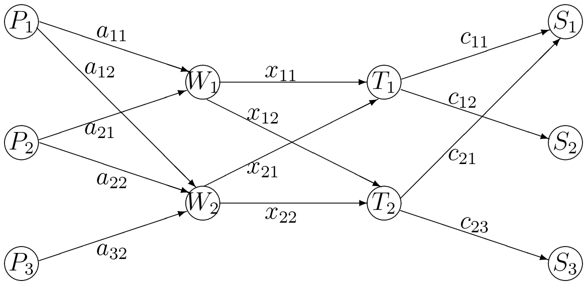

Fig. 1. Transport diagram.

Denote by b_{ik} the time scheduled to transport products from place P_i to shop S_k. To ensure the transportation for products made in P_1 to shops $S_1,\ S_2$, and S_3, the following equations must be satisfied:

$$\max\{a_{11}+x_{11}+c_{11},\ a_{11}+x_{12}+c_{21},\ a_{12}+x_{21}+c_{11},\ a_{12}+x_{22}+c_{21}\}=b_{11},$$
$$\max\{a_{11}+x_{11}+c_{12},\ a_{12}+x_{21}+c_{12}\}=b_{12},$$
$$\max\{a_{11}+x_{12}+c_{23},\ a_{12}+x_{22}+c_{23}\}=b_{13}.$$

Similar equalities must be satisfied to arrange the transportation for all products from P_2 and P_3 to shops $S_1,\ S_2$ and S_3.

In general, let us suppose that there are m places $P_1, P_2, \ldots, P_m$, n warehouses $W_1, W_2, \ldots, W_n$, s trucks $T_1, T_2, \ldots, T_s$, and r shops $S_1, S_2, \ldots, S_r$. If there is no connection from P_i to W_j (from T_l to S_k), we put $a_{ij}=-\infty$ ($c_{lk}=-\infty$). Let us denote the following index sets $M=\{1,2,\ldots,m\}$, $N=\{1,2,\ldots,n\}$, $R=\{1,2,\ldots,r\}$, and $S=\{1,2,\ldots,s\}$.

We would like to determine times x_{jl} for any $j\in N$ and for any $l\in S$ such that the maximum of total transport time from P_i to S_k is equal to a given number b_{ik} for any $i\in M$ and for any $k\in R$. It leads to solving the system of equations of the form

$$\max_{j\in N,\, l\in S}\{a_{ij}+x_{jl}+c_{lk}\}=\ b_{ik}. \tag{1}$$

2. Preliminaries

Max-plus algebra is the triple $(\overline{\mathbb{R}}, \oplus, \otimes)$, where

$$\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty\},\ \ a\oplus b=\max\{a,b\} \text{ and } a\otimes b=a+b.$$

The set of all $m\times n$ *matrices* over $\overline{\mathbb{R}}$ is denoted by $\overline{\mathbb{R}}(m,n)$ and the set of all column n-*vectors* over $\overline{\mathbb{R}}$ by $\overline{\mathbb{R}}(n)$.

Operations $\oplus$ and $\otimes$ are extended to matrices and vectors in the same way as in the classical algebra. We consider the *ordering* $\le$ on the sets $\overline{\mathbb{R}}(m,n)$ and $\overline{\mathbb{R}}(n)$ defined as follows:

- for $A, C\in\overline{\mathbb{R}}(m,n):\ A\le C$ if $a_{ij}\le c_{ij}$ for each $i\in M$ and for each $j\in N$,
- for $x, y\in\overline{\mathbb{R}}(n):\ x\le y$ if $x_j\le y_j$ for each $j\in N$.

We will use the *monotonicity of* $\otimes$, which means that for each $A, C\in\overline{\mathbb{R}}(m,n)$ and for each $B,\ D\in\overline{\mathbb{R}}(n,s)$ the implication

$$\text{if } A\le C \text{ and } B\le D \text{ then } A\otimes B\le C\otimes D$$

holds. Let $A\in\overline{\mathbb{R}}(m,n)$ and $b\in\overline{\mathbb{R}}(m)$. We can write the system of max-plus linear equations in the matrix form

$$A\otimes x=b. \tag{2}$$

It is known (see [2,15]) that system (2) is solvable if and only if the vector $x^*(A,b)$, defined by

$$x^*_j(A,b)=\min_{i\in M}\{b_i-a_{ij}\} \tag{3}$$

for any $j\in N$, where $\min\emptyset=I$, is its solution. The vector $x^*(A,b)$ is called a *principal solution* of system (2).

3. Matrix equations

Let $A=(a_{ij})\in\overline{\mathbb{R}}(m,n)$ and $B=(b_{kl})\in\overline{\mathbb{R}}(r,s)$ be given. The *tensor product* of A and B is the following matrix of size $mr\times ns$:

$$A\boxtimes B=\begin{pmatrix} A\otimes b_{11} & A\otimes b_{12} & \ldots & A\otimes b_{1s}\\ A\otimes b_{21} & A\otimes b_{22} & \ldots & A\otimes b_{2s}\\ \ldots & \ldots & \ldots & \ldots\\ A\otimes b_{r1} & A\otimes b_{r2} & \ldots & A\otimes b_{rs}\end{pmatrix},$$

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