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Note

Extremal bipartite graphs of given connectivity with respect to matching energy[☆]

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ABSTRACT

The matching energy of a graph G is $ME(G) = \frac{2}{\pi}\int_0^{\infty}\frac{1}{x^2}\ln\left[\sum_{k\geq 0} m(G,k)x^{2k}\right]dx$, and the Hosoya index of G is $Z(G) = \sum_{k\geq 0} m(G,k)$, where $m(G,k)$ is the number of k-matchings in G. In this note, we first determine the maximum values of $m(G,k)$ in all connected bipartite graphs with n vertices and a given connectivity. And then we determine the maximum matching energy (resp. Hosoya index) among all connected bipartite graphs with n vertices and a given (edge) connectivity and characterize the corresponding extremal graphs.

1. Introduction

Throughout this paper, we consider only simple and connected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, we denote by $N_G(v)$ the neighborhood of v in G. $d_G(v) = |N_G(v)|$ is called the degree of v in G. In particular, let $\delta(G) = \min\{d_G(v)|v \in V(G)\}$. For $S \subseteq V(G)$, the induced subgraph of G is denoted by $G[S]$. For $X \subset V(G)$, let $G - X$ be the graph formed from G by deleting the vertices in X and the edges incident with them. Let $G + e$ denote the graph obtained from G by adding a fresh edge $e \notin E(G)$ (i.e., e belongs to the complement of G).

A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. If G contains every edge joining a vertex of X with a vertex of Y, then G is a complete bipartite graph and is denoted by $K_{p,q}$, where $p = |X|$ and $q = |Y|$.

A cut vertex (edge) of a graph is a vertex (an edge) whose removal increases the number of components of the graph. A vertex (an edge) cut of a graph is a set of vertices (edges) whose removal disconnects the graph. The vertex connectivity $\kappa(G)$ (respectively, the edge connectivity $\kappa'(G)$) of a graph G is the minimum number of vertices (respectively, the minimum number of edges) whose deletion yields the resulting graph disconnected or a singleton. It is well known [1] that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ for any graph G.

A matching in a graph is a set of pairwise nonadjacent edges. If M is a matching, the two ends of each edge of M are said to be matched under M, and each vertex incident with an edge of M is said to be covered by M. By $m(G,k)$ we denote the number of k-matchings (i.e., the number of k-element independent edge sets) of the graph G. Specifically, $m(G,0) = 1$, $m(G,1) = |E(G)|$ and $m(G,k) = 0$ for $k > n/2$, where $n = |V(G)|$ is the number of vertices in G.

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The Hosoya index $Z(G)$ is defined as the total number of the matchings [14,15]. That is

$$Z(G) = m(G, 0) + m(G, 1) + m(G, 2) + \cdots + m(G, \lfloor n/2 \rfloor). \quad (1)$$

The matching polynomial of a graph G with order n is defined as [10,12]

$$\alpha(G, x) = \sum_{k \geq 0} (-1)^k m(G, k) x^{n-2k}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the adjacent matrix of a graph G. The energy [11] of G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

An important result in the theory of graph energy is the Coulson-type integral formula [11]

$$E(T) = \frac{2}{\pi} \int_0^\infty \frac{1}{x^2} \ln\left[\sum_{k \geq 0} m(T, k) x^{2k}\right] \mathrm{d}x. \quad (2)$$

In fact, the right hand side of formula (2) is well defined for any graph. Gutman and Wagner [13] considered it also for cycle-containing graphs, and the quantity is called the matching energy $ME(G)$ of a graph G. It is defined as

$$ME(G) = \frac{2}{\pi} \int_0^\infty \frac{1}{x^2} \ln\left[\sum_{k \geq 0} m(G, k) x^{2k}\right] \mathrm{d}x, \quad (3)$$

and it is also known that

$$ME(G) = \sum_{i=1}^{n} |\mu_i|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the zeros of its matching polynomial.

In [13], Gutman and Wagner pointed out that the matching energy is a quantity of relevance for chemical applications. Moreover, they arrived at the simple relation $TER(G) = E(G) - ME(G)$, where $TRE(G)$ denotes the topological resonance energy of the molecular graph G.

The quasi-order $\succeq$, defined by $G \succeq H \Leftrightarrow m(G, k) \geq m(H, k)$ for all $k = 0, 1, 2, \ldots, \lfloor n/2 \rfloor$. If $G \succeq H$ and there exists some k such that $m(G, k) > m(H, k)$, then we write $G \succ H$. The quasi-order has been proved fundamental in the study of the Hosoya index and the matching energy. From (1) and (3), it is clear that $G \succ H$ implies $ME(G) > ME(H)$ and $Z(G) > Z(H)$.

Lemma 1.1 (*[10,12]*). *Let G be a graph.*

(i) *If $uv \in E(G)$, then $m(G, k) = m(G - uv, k) + m(G - u - v, k - 1)$;*
(ii) *If $u \in V(G)$, then $m(G, k) = m(G - u, k) + \sum_{v \in N(u)} m(G - u - v, k - 1)$.*

Lemma 1.1(i) implies that

Lemma 1.2. *Let G be a graph of order n. Then for any edge $e \notin E(G)$, $m(G + e, k) \geq m(G, k)$ for $k = 0, 1, \ldots, \lfloor n/2 \rfloor$, and $m(G + e, k) > m(G, k)$ for at least one $0 < k \leq \lfloor n/2 \rfloor$, i.e., $G + e \succ G$.*

From Lemma 1.2 and the definitions of matching energy and Hosoya index, we have

Lemma 1.3 (*[13,25]*). *Let G be a graph of order n. Then for any edge $e \notin E(G)$, $ME(G + e) > ME(G)$ and $Z(G + e) > Z(G)$.*

The increased/decreased property of adding new edges has been successfully applied in the study of the extremal values of many graph invariants over a significant class of graphs, see [2,3,21,23,25].

Over the past few years, the extremal problem of matching energy and Hosoya index has attracted considerable attention: the extremal graphs of matching energy and Hosoya index have been studied for unicyclic graphs [6,9], for bicyclic graphs [5,18], for t-apex trees [26], for random polyphenyl chain [16,17], for random graphs [8], for graphs with given connectivity [21,25], for (n, m)-graphs with given matching number [24], for bipartite graph with given matching number [2] and for many others [4,7,18–20,22].

In this paper, we focus our attentions on bipartite graphs with given connectivity and the extremal bipartite graphs with respect to matching energy and Hosoya index are determined.

2. Main results

In this section, we will characterize the extremal graphs with the maximum matching energy and the maximum Hosoya index among all connected bipartite graphs of order n with given vertex (edge) connectivity.

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