Discrete Applied Mathematics 239 (2018) 206–212

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

On the sum of the squares of all distances in bipartite graphs with given connectivity☆

Xianya Geng *, Hongjin Zhao

Department of Mathematics and Physics, Anhui University of Science and Technology, Huainan 232001, PR China

ARTICLE INFO

Article history:
Received 9 July 2016
Received in revised form 6 December 2017
Accepted 7 December 2017
Available online 1 February 2018

Keywords:
bipartite graph
Vertex connectivity
Edge connectivity

ABSTRACT

Denote the sum of squares of all distances between all pairs of vertices in G by $S(G)$. In this paper, sharp bounds on the $S(G)$ are determined for several classes of connected bipartite graphs. All the extremal graphs having the minimal $S(G)$ in the class of all connected n-vertex bipartite graphs with a given vertex connectivity (resp. edge-connectivity) can be identified.

1. Introduction

In this paper, we only consider connected, simple and undirected graphs and assume that all graphs are connected, and refer to Bondy and Murty [2] for notation and terminologies used but not defined here.

Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G. we will use $G - v$, $G - uv$ to denote the graph that raises from G by deleting the vertex $v \in V_G$ or edge $uv \in E_G$, respectively (this notation is naturally extended if more than one vertex or edge is deleted). Similarly, $G + uv$ is a graph that arises from G by adding an edge $uv \notin E_G$, where $u, v \in V(G)$. For $v \in V_G$, we denote the neighborhood and the degree of v by $N_G(v)$ ($N(v)$ for short) and $d(v)$, $d(v) = |N_G(v)|$.

Recall that G is called *k-connected* if $|G| > k$ and is $G - Z$ is connected for every set $Z \subseteq V_G$ with $|Z| < k$. The greatest integer k such that G is k-connected is the connectivity $k(G)$ of G. Thus, $k(G) = 0$ if and only if G is disconnected or K_1, and $k(K_1) = n - 1$ for all $n \geq 1$.

Analogously, if $|G| > 1$ and $G - N$ is connected for every set $N \subseteq E_G$ of fewer than l edges, then G is called *l-edge-connected*. The greatest integer l such that G is l-connected is the *edge-connectivity* $k'(G)$ of G. In particular, $k'(G) = 0$ if G is disconnected.

A bipartite graph G is a simple graph, whose vertex set V_G can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. A bipartite graph in which every two vertices from different partition classes are adjacent is called complete, which is denoted by $K_{m,n}$, where $m = |V_1|$, $n = |V_2|$.

For two vertices $u, v \in G$ ($u \neq v$), the distance $d(u, v)$ between vertices u and v in G is the number of edges in a shortest path joining them. The distance of a vertex $u \in V(G)$, denoted by $L_G(u)$, is the sum of the squares of all distances from u in G.

Let $\mathscr{C}_n^s$ (resp. $\mathscr{D}_n^t$) be the class of all n-vertex bipartite graphs with connectivity s (resp. edge-connectivity t).

Let $S = S(G)$ be the sum of squares of distances between all pairs of vertices of G, which is denoted by

$$S = S(G) = \sum_{u,v \in V_G} d_G^2(u, v) = \frac{1}{2} \sum_{v \in V_G} L_G(v)$$

☆ Supported by National Natural Science Foundation of China (11401008, 61672001, 61572035, 61402011) and China Postdoctoral Science Foundation (2016M592030).

* Corresponding author.
E-mail address: gengxianya@sina.com (X. Geng).

https://doi.org/10.1016/j.dam.2017.12.008

This quantity was introduced by Mustapha Aouchich and Pierre Hansen in [1] and has been extensively studied in the monograph. Recently, $S(G)$ is applied to the research of distance spectral radius. Zhou and Trinajstić [17] proved an upper bound using the order n in addition to the sum of the squares of the distances $S(G)$, see [16,18]. They also proved a lower bound on the distance spectral radius of a graph using only $S(G)$. As a continuance of it, in this paper, we determine sharp bounds on $S(G)$ for several classes of connected bipartite graphs. For surveys and some up-to-date papers related to Wiener index of trees and line graphs, see [5,7,9–13,15] and [3,4,6,8,14], respectively.

In this paper we study the quantity S in the case of n-vertex bipartite graphs, which is an important class of graphs in graph theory. Based on the structure of bipartite graphs, sharp bounds on S among $\mathscr{C}_n^s$ (resp. $\mathscr{D}_n^t$) are determined. The corresponding extremal graphs are identified, respectively.

Further on we need the following lemma, which is the direct consequence of the definition of S.

Lemma 1.1. *Let G be a connected graph of order n and not isomorphic to K_n. Then for each edge $e \in \overline{G}$, $S(G) > S(G+e)$.*

2. The graph with minimum $S(G)$ among $\mathscr{C}_n^s$ (resp. $\mathscr{D}_n^t$)

In this section, we determine the sharp lower bound on the sum of all distances of graphs among $\mathscr{C}_n^s$ and $\mathscr{D}_n^t$, respectively.

In $K_{p,q}$, we assume that $p \geq q$ and by $K_{p,0}, p \geq 1$ we mean pK_1. We define two graphs $O_s \vee_1 (K_{n_1,n_2} \cup K_{m_1,m_2})$ and $O_s \vee_2 (K_{n_1,n_2} \cup K_{m_1,m_2})$, where $\cup$ is the union of two graphs, $O_s (s \geq 1)$ is an empty graph of order s and $\vee_1$ is a graph operation that joins all the vertices in O_s to the vertices belonging to the partitions of cardinality n_1 in K_{n_1,n_2} and m_1 in K_{m_1,m_2}, respectively; whereas, $\vee_2$ is a graph operation that joins all the vertices in O_s to the vertices belonging to the partitions of cardinality n_2 in K_{n_1,n_2} and m_2 in K_{m_1,m_2}, respectively. Note that $\vee_2$ is defined only when $n_2 \geq 1$ and $m_2 \geq 1$.

Theorem 2.1. *If $3p-3q-3s<2$ and $p \geq s$, then $S\big(O_s \vee_1 (K_1 \cup K_{p,q})\big) > S\big(O_s \vee_1 (K_1 \cup K_{p+1,q-1})\big)$.*

Proof. Let us denote $S\big(O_s \vee_1 (K_1 \cup K_{p,q})\big)$ by G and $S\big(O_s \vee_1 (K_1 \cup K_{p+1,q-1})\big)$ by G'. Here G and G' are depicted in Fig. 1. We partition $V_G = V_{G'}$ into $\{v\} \cup C \cup A \cup B \cup \{b_q\}$, where $C = \{c_1, c_2, \ldots, c_s\}$, $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_{q-1}\}$. Note that

$$L_G(a) = L_{G'}(a) - 3 \quad \text{for any } a \in A; \; L_G(b) = L_{G'}(b) + 3 \quad \text{for any } b \in B;$$
$$L_G(c) = L_{G'}(c) + 3 \quad \text{for any } c \in C; \; L_G(b_q) = L_{G'}(b_q) - 3p + 3s + 3q + 2;$$
$$L_G(v) = L_{G'}(v) + 5.$$

Hence, this gives

$$\begin{aligned} S(G) - S(G') &= \frac{1}{2}\left(\sum_{v \in V_G} L_G(v) - \sum_{v \in V_{G'}} L_{G'}(v)\right) \\ &= \frac{1}{2}\left(\sum_{a \in A}(L_G(a) - L_{G'}(a)) + \sum_{b \in B}(L_G(b) - L_{G'}(b)) + \sum_{c \in C}(L_G(c) - L_{G'}(c))\right) \\ &\quad + \frac{1}{2}\Big(L_G(v) - L_{G'}(v) + L_G(b_q) - L_{G'}(b_q)\Big) \\ &= \frac{1}{2}\big[-3p + 3(q-1) + 3s\big] + \frac{1}{2}\big[-3p + 3q + 3s + 7\big] \\ &= -3p + 3q + 3s + 2 > 0 \end{aligned}$$

This completes the proof. □

The following result is the direct consequence of the above lemma.

Corollary 2.2. *If $q \geq 1$, then $S\big(O_s \vee_2 (K_1 \cup K_{p,q})\big) \geq S\big(O_s \vee_1 (K_1 \cup K_{p,q})\big)$. The equality holds only when $p = q$.*

Lemma 2.3. *If $3q + 3s + 8 < 3p$, then $S\big(O_s \vee_1 (K_1 \cup K_{p,q})\big) > S\big(O_s \vee_1 (K_1 \cup K_{p-1,q+1})\big)$.*

Proof. Let us denote $S\big(O_s \vee_1 (K_1 \cup K_{p,q})\big)$ by G and $S\big(O_s \vee_1 (K_1 \cup K_{p-1,q+1})\big)$ by G'. We partition $V_G = V_{G'}$ into $\{v\} \cup C \cup A \cup B \cup \{u\}$, where $C = \{c_1, c_2, \ldots, c_s\}$, $A = \{a_1, a_2, \ldots, a_{p-1}\}$ and $B = \{b_1, b_2, \ldots, b_q\}$ (see Fig. 2).
Note that

$$L_G(a) = L_{G'}(a) + 3 \quad \text{for any } a \in A; \; L_G(b) = L_{G'}(b) - 3 \quad \text{for any } a \in B;$$
$$L_G(c) = L_{G'}(c) - 3 \quad \text{for any } a \in C; \; L_G(v) = L_{G'}(v) - 5;$$
$$L_G(b_q) = L_{G'}(b_q) + 3p - 3s - 3q - 8.$$