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Note

Lower bounds for the Laplacian energy of bipartite graphs

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ABSTRACT

Given a simple undirected graph $G = (V, E)$ with n vertices, if for the largest eigenvalue of its Laplacian matrix λ_1 there exists a lower bound $\lambda_1 \geq \alpha \geq d_G \frac{n}{n-1}$, then we have that its Laplacian energy satisfies

$$LE(G) \geq \max\{2d_G, 2(\alpha - d_G)\},$$

where $d_G = \frac{d_1+\cdots d_n}{n}$ is the average degree of G. This generic lower bound, obtained with the majorization technique, allows us to obtain two lower bounds for $LE(G)$ which are valid for any connected bipartite graph, and for which the equalities are attained by $K_{\frac{n}{2},\frac{n}{2}}$ and S_n.

1. Introduction

Let $G = (V, E)$ be a finite simple undirected graph with vertex set $V = \{1, 2, \ldots, n\}$ and degrees $d_1 \geq d_2 \geq \cdots \geq d_n$, and $d_G = \frac{2|E|}{n}$ the *average degree*. If $\mathbf{A}$ is the adjacency matrix of G and $\mathbf{D}$ is the diagonal matrix whose entries are the degrees of the graph, one defines the Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$. There are several descriptors in Mathematical Chemistry defined in terms of these eigenvalues; among them the *Laplacian energy* is defined in [4] as

$$LE(G) = \sum_{i=1}^{n} |\lambda_i - d_G|.$$

In [4] it was shown for an arbitrary G that

$$LE(G) \geq 2\sqrt{|E| + \frac{1}{2}\sum_{i=1}^{n}(d_i - d_G)^2}, \tag{1}$$

and the equality is attained by the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. In [8] it was established for an arbitrary G that

$$LE(G) \geq \sum_{i=1}^{n} |d_i - d_G|, \tag{2}$$

an inequality that was slightly improved for connected graphs in [7] as

$$LE(G) \geq 2 + \sum_{i=1}^{n} |d_i - d_G|, \tag{3}$$

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and where the equality is attained by the star graph S_n. Also, in [10] it was proven that

$$LE(G) \geq 2d_G, \tag{4}$$

for an arbitrary G, and where the equality is attained by any regular complete k-partite graph, for $1 \leq k \leq n$. Finally, in [2] it was shown that for a connected G we have

$$LE(G) \geq 2(d_1 + 1 - d_G), \tag{5}$$

where the equality is attained by S_n, and more generally, it was argued that if σ $(1 \leq \sigma \leq n-1)$ is the largest integer such that $\lambda_\sigma \geq d_G$, one has

$$LE(G) \geq 2(\sum_{j=1}^{\sigma} d_j + 1 - \sigma d_G). \tag{6}$$

It is worth to mention that for d-regular graphs, the bounds (1) through (6) become $\sqrt{2nd}$, 0, 2, 2d, 2 and 2, respectively, pointing out that they are not comparable.

In this note, using the majorization technique, we find a new general lower bound for the Laplacian energy of graphs, not necessarily connected, satisfying a condition on the largest eigenvalue of their Laplace matrix, and as corollaries we obtain two new non comparable lower bounds for the Laplacian energy of connected bipartite graphs.

The majorization technique can be summarized as follows: given two n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ with $x_1 \geq x_2 \ldots \geq x_n$ and $y_1 \geq y_2 \ldots y_n$, we say that $\mathbf{x}$ majorizes $\mathbf{y}$ and write $\mathbf{x} \succ \mathbf{y}$ in case

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \tag{7}$$

for $1 \leq k \leq n-1$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{8}$$

A Schur-convex function $\Phi : \mathbb{R} \to \mathbb{R}$ keeps the majorization inequality, that is, if Φ is Schur-convex then $\mathbf{x} \succ \mathbf{y}$ implies $\Phi(\mathbf{x}) \geq \Phi(\mathbf{y})$. Likewise, a Schur-concave function reverses the inequality: for this type of function $\mathbf{x} \succ \mathbf{y}$ implies $\Phi(\mathbf{x}) \leq \Phi(\mathbf{y})$. A simple way to construct a Schur-convex (resp. Schur-concave) function is to consider

$$\Phi(\mathbf{x}) = \sum_{i=1}^{n} f(x_i),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a convex (resp. concave) one-dimensional real function. For more details on majorization the reader is referred to [6].

2. Lower bounds

The usual approach of the majorization technique presented in the introduction is the search on a certain subset $S \subset \mathbb{R}^n$, of its maximal and minimal elements, $\mathbf{x}^*$ and $\mathbf{x}_*$, respectively, which satisfy

$$\mathbf{x}^* \succ \mathbf{x}$$

and

$$\mathbf{x} \succ \mathbf{x}_*,$$

respectively, for all $\mathbf{x} \in S$. If that is the case then

$$\Phi(\mathbf{x}^*) \geq \Phi(\mathbf{x})$$

and

$$\Phi(\mathbf{x}) \geq \Phi(\mathbf{x}_*), \tag{9}$$

for all $\mathbf{x} \in S$. We will use a known result that identifies $\mathbf{x}_*$ for particular subsets $S \subset \mathbb{R}^n$ and apply (9), when Φ is replaced with the appropriate descriptor, in order to find the lower bounds desired. Specifically, we note that the one-dimensional real function $f(x) = |x - d_G|$ is convex and therefore the descriptor $LE(G)$ is a Schur-convex function if we replace the eigenvalues λ_i in its definition with arbitrary real numbers x_i. Now we can show the following preliminary result.

Proposition 1. *For any G, not necessarily connected, if there is α such that $\lambda_1 \geq \alpha \geq d_G \frac{n}{n-1}$ then we have*

$$LE(G) \geq \max\{2d_G, 2(\alpha - d_G)\}. \tag{10}$$