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The shape of node reliability

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ABSTRACT

Given a graph G whose edges are perfectly reliable and whose nodes each operate independently with probability $p \in [0, 1]$, the *node reliability* of G is the probability that at least one node is operational and that the operational nodes can all communicate in the subgraph that they induce. We study analytic properties of the node reliability on the interval $[0, 1]$ including monotonicity, concavity, and fixed points. Our results show a stark contrast between this model of network robustness and models that arise from coherent set systems (including all-terminal, two-terminal and K-terminal reliability).

1. Introduction

There are a number of models of probabilistic network reliability based on the premise that in a (finite, undirected) graph, the nodes are always operational, but edges are independently operational with probability p. Failure in such a system may be due to random failures of components in the links joining nodes. The well-known *all-terminal reliability* asks for the probability that all vertices can communicate with one another; that is, the probability that at least a spanning tree is operational. This model generalizes to K-*terminal reliability*, which asks the probability that all vertices in some particular subset K can communicate with one another. We call the vertices in K the *target nodes*, with the target nodes ranging from two particular vertices in the well-studied *two-terminal reliability* to the entire vertex set for all-terminal reliability. An excellent survey of these measures can be found in [4].

These models of reliability fit under the umbrella of *coherence*. Let X be a finite ground set. A *coherent set system* $\mathcal{S}$ on X is a subset of $\mathcal{P}(X)$, the powerset of X, that satisfies the following conditions:

(i) if $S_1 \in \mathcal{S}$ and $S_1 \subseteq S_2 \subseteq X$ then $S_2 \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed under taking supersets in X),
(ii) $\emptyset \notin \mathcal{S}$, and
(iii) $X \in \mathcal{S}$.

The *order* of $\mathcal{S}$ is the cardinality of the ground set X. We think of the elements of X as components of a system that either operate or fail, hence we call the sets in $\mathcal{S}$ the *operational states*. Coherence is then the natural property that if we start with an operational state, then making any number of failed components operational will not result in a failed state. Let X have cardinality n and suppose that each element of X is independently operational with probability $p \in (0, 1)$. The *reliability* of

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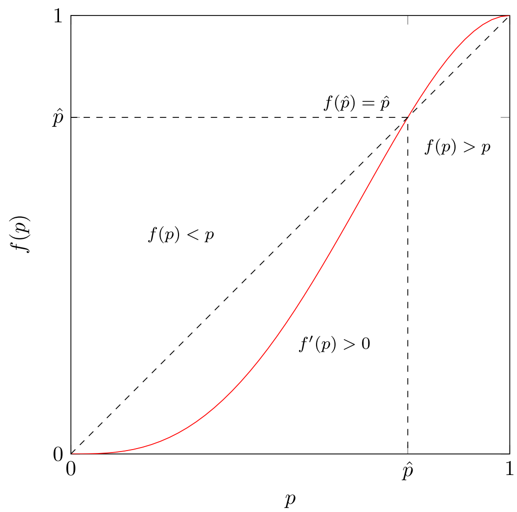

Fig. 1. A plot of an S-shaped function f.

coherent set system $\mathcal{S}$ on X, denoted $\text{Rel}(\mathcal{S}; p)$, is the probability that the set of operational elements of X is in $\mathcal{S}$; that is,

$$\text{Rel}(\mathcal{S}; p) = \sum_{S \in \mathcal{S}} p^{|S|}(1-p)^{n-|S|} \tag{1}$$

$$= \sum_{i=0}^{n} N_i p^i (1-p)^{n-i} \tag{2}$$

where N_i is the number of operational states of order i for each $i \in \{1, \ldots, n\}$. There are obvious relevant coherent set systems underlying each of the network models introduced earlier, all on the edge set of the graph; in general for K-terminal reliability, the operational states are those edge subsets that connect all vertices of K. This collection of subsets is easily seen to be closed under taking supersets in the edge set, as adding edges cannot possibly disconnect an already connected graph.

Birnbaum, Esary, and Saunders achieved several significant results in [1] that describe the general shape of a coherent reliability polynomial (that is, the reliability of any coherent set system) on the interval $[0, 1]$.

- The reliability of any coherent set system $\mathcal{S}$ is strictly increasing on $(0, 1)$ and satisfies $\text{Rel}(\mathcal{S}; 0) = 0$ and $\text{Rel}(\mathcal{S}; 1) = 1$.
- The reliability of any coherent set system of order at least 2 has at most one fixed point in $(0, 1)$.
- When written in the form of (2), the reliability of any coherent set system $\mathcal{S}$ with $N_1 = 0$ and $N_{n-1} = n$ has exactly one fixed point $\hat{p} \in (0, 1)$. Further, $\text{Rel}(\mathcal{S}; p) < p$ for $p \in (0, \hat{p})$ and $\text{Rel}(\mathcal{S}; p) > p$ for $p \in (\hat{p}, 1)$.

Note that the first result listed above implies that the all-terminal reliability of any graph with at least two vertices is strictly increasing on $(0, 1)$ and passes through the points $(0, 0)$ and $(1, 1)$, and the second implies that the all-terminal reliability of any graph with at least two edges has at most one fixed point in $(0, 1)$. The conditions $N_1 = 0$ and $N_{n-1} = n$ of the third result listed above mean simply that the system fails whenever at most one component is operational, and that the system is operational whenever at most one component fails, respectively. For all-terminal reliability, these conditions are satisfied if and only if the graph lies on at least 3 vertices and is 2-edge-connected (i.e. has no bridges). For all graphs satisfying these fairly weak conditions, we get a nice general description of the shape of the all-terminal reliability polynomial on $[0, 1]$. Further, it is easily verified that the all-terminal reliability of any graph on at least 3 vertices that is not 2-edge-connected lies below the identity function for $p \in (0, 1)$. So the results of Birnbaum et al. give us a very good idea of the shape of any all-terminal reliability polynomial on the interval $[0, 1]$.

These findings led to the definition of an *S-shaped* (or *sigmoid shaped*) curve. A typical S-shaped curve is shown in Fig. 1. In general, a function f is called *S-shaped* on $[0, 1]$ if it satisfies the following:

- $f(0) = 0$ and $f(1) = 1$,
- $f'(p) > 0$ for $p \in (0, 1)$,
- $f(p)$ has a unique fixed point $\hat{p} \in (0, 1)$, and
- $f(p) < p$ for $p \in (0, \hat{p})$ and $f(p) > p$ for $p \in (\hat{p}, 1)$.

With this new terminology, the results of Birnbaum et al. listed above say that coherent reliability polynomials are S-shaped under fairly weak conditions.

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