Discrete Applied Mathematics 238 (2018) 70–76

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Sufficient conditions for a balanced bipartite digraph to be even pancyclic

Samvel Kh. Darbinyan

Institute for Informatics and Automation Problems, Armenian National Academy of Sciences, Armenia

ARTICLE INFO

Article history:
Received 28 March 2017
Received in revised form 7 December 2017
Accepted 8 December 2017
Available online 5 January 2018

Keywords:
Digraphs
Cycles
Hamiltonian cycles
Bipartite balanced digraph
Pancyclic
Even pancyclic
Longest non-Hamiltonian cycles

ABSTRACT

Let D be a strongly connected balanced bipartite directed graph of order $2a \geq 8$. In this note we prove:

(i). *If D contains a cycle of length $2a-2 \geq 6$ and $\max\{d(x), d(y)\} \geq 2a-2$ for every pair of vertices $\{x, y\}$ with a common out-neighbour, then for every k, $1 \leq k \leq a-1$, D contains a cycle of length $2k$.*

(ii). *If D is not a directed cycle and $\max\{d(x), d(y)\} \geq 2a-1$ for every pair of vertices $\{x, y\}$ with a common out-neighbour, then for every k, $1 \leq k \leq a$, D contains a cycle of length $2k$ unless D is isomorphic to a certain digraph of order eight which we specify.*

1. Introduction

We consider directed graphs (digraphs) in the sense of [3]. A cycle is called Hamiltonian if it includes all the vertices of D. A digraph D is Hamiltonian if it contains a Hamiltonian cycle.

Let us recall the following well-known degree conditions (Theorems 1.1–1.4) that guarantee that a digraph is Hamiltonian.

Theorem 1.1 (*Ghouila-Houri [12]*). *Let D be a strongly connected digraph of order $n \geq 3$. If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is Hamiltonian.*

Theorem 1.2 (*Woodall [21]*). *Let D be a digraph of order $n \geq 3$. If $d^+(x) + d^-(y) \geq n$ for all pairs of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 1.3 (*Meyniel [16]*). *Let D be a strongly connected digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D is Hamiltonian.*

Notice that Meyniel's theorem is a generalization of Ghouila-Houri's and Woodall's theorems. For a short proof of Theorem 1.3, see [6]. Thomassen [19] (for $n = 2m+1$) and Darbinyan [9] (for $n = 2m$) proved the following theorem.

Theorem 1.4 (*Thomassen [19], Darbinyan [9]*). *If D is a digraph of order $n \geq 5$ with minimum degree at least $n-1$ and with minimum semi-degree at least $n/2-1$. Then D is Hamiltonian or belongs to a non-empty finite family of non-Hamiltonian digraphs, which are characterized.*

E-mail address: samdarbin@ipia.sci.am.

https://doi.org/10.1016/j.dam.2017.12.013
0166-218X/

A digraph D is pancyclic if it contains cycles of every length k, $3 \le k \le n$, where n is the order of D. Bondy suggested (see [7] by Chvátal) the following interesting metaconjecture:

Metaconjecture (Bondy). *Almost any non-trivial condition of a undirected graph (digraph) which implies that the graph (digraph) is Hamiltonian also implies that the undirected graph (digraph) is pancyclic. (There may be a "simple" family of exceptional graphs (digraphs)).*

There are various sufficient conditions for a digraph (undirected graph) to be Hamiltonian are also sufficient for the digraph (undirected graph) to be pancyclic. In particular, in [17,15,18,8,10] it was shown that if a digraph D satisfies the condition one of Theorems 1.1–1.4, respectively, then D also is pancyclic (unless some extremal cases which are characterized).

Characterizations of even pancyclic bipartite tournaments was given in [5,22]. A characterization of pancyclic ordinary k-partite ($k \ge 3$) tournaments (respectively, ordinary complete k-partite ($k \ge 3$) digraphs) was established in [13] (respectively, [14]). Amar and Manoussakis [1] gave several sufficient conditions on the half-degrees of a bipartite digraph for the existence of cycles and paths of various lengths.

Each of Theorems 1.1–1.4 imposes a degree condition on all pairs of non-adjacent vertices (or on all vertices). In [2,4] it was shown some sufficient conditions for hamiltonicity of digraphs in which the degree conditions require only for some pairs of nonadjacent vertices. Let us recall of them only the following theorem.

Theorem 1.5 (*Bang-Jensen, Gutin, Li [4]*)**.** *Let D be a strongly connected digraph of order $n \ge 2$. Suppose that $min\{d(x), d(y)\} \ge n-1$ and $d(x) + d(y) \ge 2n-1$ for any pair of non-adjacent vertices x, y with a common in-neighbour. Then D is Hamiltonian.*

Let x, y be distinct vertices in D. $\{x, y\}$ dominates a vertex z if $x \to z$ and $y \to z$; in this case, we call the pair $\{x, y\}$ dominating.

An analogue of Theorem 1.5 for bipartite digraphs was given by Wang [20], and recently strengthened by the author [11].

Theorem 1.6 (*Wang [20]*)**.** *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \ge 1$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \ge 2a-1$ and $d(y) \ge a+1$ or $d(y) \ge 2a-1$ and $d(x) \ge a+1$. Then D is Hamiltonian.*

Before stating the next theorem we need to define a digraph of order eight.

Definition. Let $D(8)$ be the bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$, and $A(D(8))$ contains exactly the arcs $y_0x_1, y_1x_0, x_2y_3, x_3y_2$ and all the arcs of the following 2-cycles: $x_i \leftrightarrow y_i$, $i \in [0, 3]$, $y_0 \leftrightarrow x_2$, $y_0 \leftrightarrow x_3$, $y_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow x_3$.

It is not difficult to check that $D(8)$ is strongly connected, $max\{d(x), d(y)\} \ge 2a-1$ for every pair of vertices $\{x, y\}$ with a common out-neighbour. It is not difficult to check that D is not Hamiltonian. Indeed, if C is a Hamiltonian cycle in $D(8)$, then C would contain the arcs x_1y_1 and x_0y_0 and therefore, the path $x_1y_1x_0y_0$ or the path $x_0y_0x_1y_1$, which is impossible since $N^-(x_0) = N^-(x_1) = \{y_0, y_1\}$.

We also need the following definition.

Definition. Let D be a balanced bipartite digraph of order $2a \ge 4$ and k be an integer. We say that D satisfies condition B_k if for every dominating pair of vertices x and y, $max\{d(x), d(y)\} \ge 2a-2+k$.

Theorem 1.7 (*Darbinyan [11]*)**.** *Let D be a strongly connected balanced bipartite digraph of order $2a$, where $a \ge 4$. If D satisfies condition B_1, i.e., $max\{d(x), d(y)\} \ge 2a-1$ for every dominating pair of vertices $\{x, y\}$, then D is Hamiltonian unless D is isomorphic to the digraph $D(8)$.*

A balanced bipartite digraph of order $2m$ is even pancyclic if it contains cycles of every length $2k$, $2 \le k \le m$.

Motivated by the Bondy's metaconjecture, it is natural to pose the following problem:

Problem. Characterize those digraphs which satisfy the conditions of Theorem 1.7 (1.6) but are not even pancyclic.

In this note we prove the following theorems.

Theorem 1.8. *Let D be a strongly connected balanced bipartite digraph of order $2a \ge 8$ with partite sets X and Y. Suppose that D is not a directed cycle and satisfies condition B_1, i.e., $max\{d(x), d(y)\} \ge 2a-1$ for every dominating pair of vertices $\{x, y\}$. Then D contains a cycle of length $2a-2$.*

Theorem 1.9. *Let D be a strongly connected balanced bipartite digraph of order $2a \ge 8$ with partite sets X and Y. If D contains a cycle of length $2a-2$ and satisfies condition B_0, i.e., $max\{d(x), d(y)\} \ge 2a-2$ for every dominating pair of vertices $\{x, y\}$, then D contains cycles of every length $2k$, $1 \le k \le a-1$.*

Theorem 1.10. *Let D be a strongly connected balanced bipartite digraph of order $2a \ge 8$ with partite sets X and Y. If D is not a directed cycle and satisfies condition B_1, i.e., $max\{d(x), d(y)\} \ge 2a-1$ for every dominating pair of vertices $\{x, y\}$, then either D contains cycles of all even lengths less than or equal to $2a$ or D is isomorphic to the digraph $D(8)$.*