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Generalised Ramsey numbers for two sets of cycles

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ABSTRACT

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two sets of cycles. We determine all generalised Ramsey numbers $R(\mathcal{C}_1, \mathcal{C}_2)$ such that $\mathcal{C}_1$ or $\mathcal{C}_2$ contains a cycle of length at most 6. This generalises previous results of Erdős, Faudree, Rosta, Rousseau, and Schelp. Furthermore, we give a conjecture for the general case. We also provide a complete classification of most $(\mathcal{C}_1, \mathcal{C}_2)$-critical graphs such that $\mathcal{C}_1$ or $\mathcal{C}_2$ contains a cycle of length at most 5. For length 4, this is an easy extension of a recent result of Wu, Sun, and Radziszowski, in which $|\mathcal{C}_1| = |\mathcal{C}_2| = 1$. For lengths 3 and 5, our results are new also in this special case.

1. Introduction

All graphs in this paper are finite, simple, and undirected. Furthermore, (G_1, G_2) and $(\mathcal{G}_1, \mathcal{G}_2)$ denote a pair of non-empty graphs and a pair of non-empty sets of non-empty graphs, respectively. Notation will generally follow [2].

Here, a *red–blue graph* is a *complete* graph with each edge coloured either red or blue. Red will always be the first colour and blue will always be the second.

The following definition is fundamental to this paper.

Definition 1.1. The *generalised Ramsey number* $R(\mathcal{G}_1, \mathcal{G}_2)$ is the least positive integer n, such that each red–blue graph on n vertices contains a red subgraph from $\mathcal{G}_1$ or a blue subgraph from $\mathcal{G}_2$.

Note that when $|\mathcal{G}_1| = |\mathcal{G}_2| = 1$, the generalised Ramsey number $R(\mathcal{G}_1, \mathcal{G}_2)$ reduces to the ordinary Ramsey number $R(G_1, G_2)$. It is easy to see that $R(\mathcal{G}_1, \mathcal{G}_2) \le R(\mathcal{H}_1, \mathcal{H}_2)$ if each $\mathcal{H}_i$ is a non-empty subset of $\mathcal{G}_i$. Thus

$$R(\mathcal{G}_1, \mathcal{G}_2) \le R(G_1, G_2)$$

if each $G_i \in \mathcal{G}_i$. In particular, $R(\mathcal{G}_1, \mathcal{G}_2)$ always exists. Clearly, $R(\mathcal{G}_1, \mathcal{G}_2) = R(\mathcal{G}_2, \mathcal{G}_1)$.

Early work on generalised Ramsey numbers for two sets of graphs include [4,5,7,12]. In [4], the case of two sets of cycles is considered, while the others deal with a set of cycles versus a complete graph. Some interesting applications of results on generalised Ramsey numbers for two sets of graphs, are computations of exact values of multicolour Ramsey numbers for cycles (see [4]) and of (ordinary) Ramsey numbers for a cycle versus a complete graph (see [7]), and a new proof of the existence of graphs with arbitrarily large girth and chromatic number (see [12]), a result due to Erdős [3].

Let G be a red–blue graph. Then G is called $(\mathcal{G}_1, \mathcal{G}_2)$-*avoiding* if G contains neither a red subgraph from $\mathcal{G}_1$ nor a blue subgraph from $\mathcal{G}_2$, and $(\mathcal{G}_1, \mathcal{G}_2)$-*critical* if, moreover, G has $R(\mathcal{G}_1, \mathcal{G}_2) - 1$ vertices. The *red subgraph* G_{red} of G is the graph $(V(G), \{e \in E(G) \mid e \text{ is red}\})$; the *blue subgraph* G_{blue} is defined analogously. When we say that G is red hamiltonian, blue bipartite, etc., we mean that G_{red} is hamiltonian, G_{blue} is bipartite, and so on. Further terminology will be introduced in Section 2.

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Throughout this paper, $(\mathcal{C}_1, \mathcal{C}_2)$ *denotes a pair of non-empty sets of cycles.* The main results can be divided into two groups: computation of generalised Ramsey numbers $R(\mathcal{C}_1, \mathcal{C}_2)$ on the one hand, and classification of $(\mathcal{C}_1, \mathcal{C}_2)$-avoiding and $(\mathcal{C}_1, \mathcal{C}_2)$-critical graphs on the other.

1.1. Previous results

We first present some results which, to the best of the author's knowledge, include all previously known exact values of generalised Ramsey numbers for two sets of cycles.

When $|\mathcal{C}_1| = |\mathcal{C}_2| = 1$, the (ordinary) Ramsey numbers $R(\mathcal{C}_1, \mathcal{C}_2)$ were determined independently by Rosta [11] and by Faudree and Schelp [6]. A new proof, simpler but still quite technical and detailed, was given by Károlyi and Rosta [10].

Theorem 1.2 *([11] and [6]). Let* $n \geq k \geq 3$. *Then*

$$R(C_n, C_k) = \begin{cases} 6 & \text{if } (n,k) \in \{(3,3),(4,4)\}, \\ n + k/2 - 1 & \text{if } n \text{ and } k \text{ are even, } n \neq 4, \\ \max(n + k/2 - 1, 2k - 1) & \text{if } n \text{ is odd and } k \text{ is even,} \\ 2n - 1 & \text{if } k \text{ is odd, } n \neq 3. \end{cases}$$

Moreover, we have the following three results of Erdős, Faudree, Rousseau, and Schelp. From the latter two of these results, we obtain Corollary 1.6.

Theorem 1.3 *([4]). Let* $n \geq k \geq 3$. *Then*

$$R(\mathcal{C}_{\text{odd}} \cap \{C_i \mid i \leq n\}, \mathcal{C}_{\text{odd}} \cap \{C_i \mid i \leq k\}) = \begin{cases} 6 & \text{if } (n,k) = (3,3), \\ 5 & \text{otherwise,} \end{cases}$$

where $\mathcal{C}_{\text{odd}}$ *denotes the set of all odd cycles.*

Theorem 1.4 *([5, Theorem 3]). For all* $n > m \geq 2$,

$$R(\{C_i \mid i \leq n\}, \{K_m\}) = \begin{cases} 2m & \text{if } m < n < 2m - 1, \\ 2m - 1 & \text{if } n \geq 2m - 1. \end{cases}$$

Theorem 1.5 *([7, Theorem 2]). For all* $n \geq 3$ *and all* $m \geq 2$,

$$R(\{C_i \mid i \geq n\}, \{K_m\}) = (n-1)(m-1) + 1.$$

Corollary 1.6. *For all* $n \geq 3$,

$$R(\{C_i \mid i \leq n\}, \{C_3\}) = \begin{cases} 6 & \text{if } n \leq 4, \\ 5 & \text{if } n \geq 5, \end{cases}$$

and

$$R(\{C_i \mid i \geq n\}, \{C_3\}) = 2n - 1.$$

Let us turn to a structural result, due to Wu, Sun, and Radziszowski [13]. In order to state it, we have to define some sets of graphs. Given a graph G, a vertex $v \in V(G)$, a subset $U \subseteq V(G)$, and an edge $e \in E(G)$, let $N(v)$, $d(v)$, $G[U]$, and $G - e$ denote the set of neighbours of v, the number of neighbours of v, the induced subgraph on U, and the graph G with the edge e deleted, respectively. We say that G has a *matching on* U if each vertex in U has degree at most 1 in $G[U]$.

Definition 1.7. Let $n \geq 6$. For each $i \in [3]$, let $\mathcal{G}_i$ be a set of graphs on $\{v\} \cup X \cup \{y\}$ with $X = \{x_1, \ldots, x_{n-2}\}$, as follows:

- $\mathcal{G}_1$ consists of the graphs with $N(v) = X$, having a matching on $X \cup \{y\}$;
- $\mathcal{G}_2$ consists of the graphs with $N(v) = X \cup \{y\}$, having a matching on $X \cup \{y\}$;
- $\mathcal{G}_3$ consists of the graphs with $N(v) = X$, $N(y) = \{x_{n-2}\}$, and $d(x_{n-2}) = 3$, having a matching on X.

See [13, Figure 1] for pictures illustrating these graph sets.

We can now state the structural result:

Theorem 1.8 *([13, Theorem 1]). Let* $n \geq 6$. *Then* G *is* (C_n, C_4)*-critical if and only if* $G_{\text{blue}} \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

1.2. Main results

Let c_i and c_i^e denote the length of the shortest cycle and the length of the shortest even cycle, respectively, in $\mathcal{C}_i$, where $c_i^e = \infty$ if $\mathcal{C}_i$ contains no even cycle. We shall later (in Section 3.1) define a number $\mathfrak{m} = \mathfrak{m}(\mathcal{C}_1, \mathcal{C}_2)$, and we shall prove that

$$\mathfrak{m} = \max\big(5, \min(c_2 + c_1^e/2 - 1, 2c_2 - 1), \min(c_1 + c_2^e/2 - 1, 2c_1 - 1)\big).$$