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List star edge coloring of sparse graphs

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ABSTRACT

A *star-edge coloring* of a graph G is a proper edge coloring such that every 2-colored connected subgraph of G is a path of length at most 3. For a graph G, let the *list star chromatic index* of G, $ch'_s(G)$, be the minimum k such that for any k-uniform list assignment L for the set of edges, G has a star-edge coloring from L. Dvořák et al. (2013) asked whether the list star chromatic index of every subcubic graph is at most 7. In Kerdjoudj et al. (2017) we proved that it is at most 8. In this paper we give a partial answer to the question of Dvořák et al. (2013) by proving that if the maximum average degree of a subcubic graph G is less than $\frac{30}{11}$ then $ch'_s(G) \leq 7$.

We consider also graphs with any maximum degree, we proved that if the maximum average degree of a graph G is less than $\frac{7}{3}$ (resp., $\frac{5}{2}$, $\frac{8}{3}$), then $ch'_s(G) \leq 2\Delta(G) - 1$ (resp., $ch'_s(G) \leq 2\Delta(G)$, $ch'_s(G) \leq 2\Delta(G) + 1$).

1. Introduction

All the graphs we consider are finite and simple. For a graph G, we denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ its vertex set, edge set, minimum degree and maximum degree, respectively.

A *proper* vertex (respectively, edge) coloring of G is an assignment of colors to the vertices (respectively, edges) of G such that no two adjacent vertices (respectively, edges) receive the same color. A *star coloring* of G is a proper vertex coloring of G such that the union of any two color classes induces a *star forest* in G, i.e. every component of this union is a star. This notion was first mentioned by Grünbaum [6] in 1973 (see also [5]). The star coloring even in the class of line graphs seems to be difficult. A convenient language for discussions of star coloring of line graphs is the language of star-edge coloring of all graphs.

A *star-edge coloring* of a graph G is a proper edge coloring such that every 2-colored connected subgraph of G is a path of length at most 3. In other words, we forbid bi-colored 4-cycles and 4-paths in G (by a k-path we mean a path with k edges). This notion is intermediate between *acyclic edge coloring*, when every 2-colored subgraph must be only acyclic, and *strong edge coloring*, when every 2-colored connected subgraph is a path of length 2. The *star chromatic index* of G, denoted by $\chi'_{st}(G)$, is the minimum number of colors needed for a star-edge coloring of G. It was first studied by Liu and Deng [8] in 2008. They proved the following upper bound.

Theorem 1.1 ([8]). *For every G with maximum degree $\Delta \geq 7$, $\chi'_{st}(G) \leq \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$.*

In [2] and later [1] it is proved as follows:

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Theorem 1.2 ([2,1]). *The star chromatic index of any tree with maximum degree Δ is at most $\Delta + \lceil \frac{\Delta-1}{2} \rceil$.*

In [3], Dvořák, Mohar and Šámal showed that even determining the star chromatic index of the complete graph K_n with n vertices is a hard problem. They gave the following bounds:

$$2n(1+o(1)) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log(n)}}}{\log n^{\frac{1}{4}}}.$$

They also studied the star chromatic index of *subcubic* graphs, that is, graphs with maximum degree at most 3. They proved that $\chi'_{st}(G) \leq 7$ for every subcubic graph G, and conjectured that $\chi'_{st}(G) \leq 6$ for every such G. A natural generalization of star-edge coloring is the list star-edge coloring. An *edge list* L for a graph G is a mapping that assigns a finite set of colors to each edge of G. Given an edge list L for a graph G, we say that G is *L-star-edge colorable* if it has a star-edge coloring c such that $c(e) \in L(e)$ for every edge $e \in E(G)$. The *list-star chromatic index*, $ch'_s(G)$, of a graph G is the minimum k such that for every edge list L for G with $|L(e)| = k$ for every $e \in E(G)$, G is L-star-edge colorable. Dvořák, Mohar and Šámal [3, Question 3] asked whether $ch'_s(G) \leq 7$ for every subcubic G.

In [7] it is proved that the following theorem holds.

Theorem 1.3. *For every subcubic graph G, $ch'_s(G) \leq 8$.*

We also gave sufficient conditions for the list-star chromatic index of a subcubic graph to be at most 5 and 6 in terms of the *maximum average degree* $\text{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|},\ H \subseteq G\right\}$.

Theorem 1.4 ([7]). *Let G be a subcubic graph.*

1. *If $mad(G) < \frac{7}{3}$ then $ch'_s(G) \leq 5$.*
2. *If $mad(G) < \frac{5}{2}$ then $ch'_s(G) \leq 6$.*

In this paper we will prove a complement for Theorem 1.4.

Theorem 1.5. *Let G be a subcubic graph. If $mad(G) < \frac{30}{11}$ then $ch'_s(G) \leq 7$.*

This a partial answer to the Question 3 of Dvořák, Mohar and Šámal [3].

We will also extend Theorem 1.4 to graphs with any maximum degree, and we give bounds for the chromatic index of sparse graphs (i.e. graphs with a small maximum average degree) in terms of the maximum degree. We prove the following theorem:

Theorem 1.6. *Let G be a graph with maximum degree Δ.*

1. *If $mad(G) < \frac{7}{3}$ then $ch'_s(G) \leq 2\Delta - 1$.*
2. *If $mad(G) < \frac{5}{2}$ then $ch'_s(G) \leq 2\Delta$.*
3. *If $mad(G) < \frac{8}{3}$ then $ch'_s(G) \leq 2\Delta + 1$.*

It is clear that the case $\Delta = 3$ relies on Theorem 1.4 in [7].

The girth of a graph G is the size of a smallest cycle in G. An easy application of Euler's formula shows that for every planar graph G with girth g satisfies $\text{mad}(G) < \frac{2g}{g-2}$. The following corollary can be derived from Theorem 1.6.

Corollary 1. *Let G be a planar graph with girth g.*

1. *If $g \geq 14$ then $ch'_s(G) \leq 2\Delta - 1$.*
2. *If $g \geq 10$ then $ch'_s(G) \leq 2\Delta$.*
3. *If $g \geq 8$ then $ch'_s(G) \leq 2\Delta + 1$.*

The structure of the paper is as follows. Before proving our results, we introduce some notations and give an useful lemma proved in [7] on extensions of partial star-edge coloring. In the following three sections we prove Theorem 1.6 and in the last section we prove Theorem 1.5.

Notations. For a graph G, we denote by $d_G(v)$ the degree of a vertex v in G and by $N_G(v)$ the set of neighbors of v in G. If G is clear from the content, we may omit the subscript. A vertex of degree k is called a k-vertex. A k^+-vertex (respectively, k^--vertex) is a vertex of degree at least k (respectively, at most k). A *k-neighbor* of a vertex v is a k-vertex adjacent to v.

For an edge coloring ϕ of a graph G and a vertex $v \in V(G)$, $\phi(v)$ denotes the set of colors used on the edges incident with v. A p-thread $xu_1u_2 \cdots u_py$ between x and y is a path such that $d(u_1) = d(u_2) = \cdots = d(u_p) = 2$. When a p-thread exists between two vertices x and y, we say that they are p-linked.

Now the useful lemma is proved in [7].