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Graphs whose Wiener index does not change when a specific vertex is removed

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ABSTRACT

The Wiener index $W(G)$ of a connected graph G is defined to be the sum of distances between all pairs of vertices in G . In 1991, Šoltés studied changes of the Wiener index caused by removing a single vertex. He posed the problem of finding all graphs G so that equality $W(G) = W(G - v)$ holds for all their vertices v . The cycle with 11 vertices is still the only known graph with this property. In this paper we study a relaxed version of this problem and find graphs which Wiener index does not change when a particular vertex v is removed. We show that there is a unicyclic graph G on n vertices with $W(G) = W(G - v)$ if and only if $n \geq 9$. Also, there is a unicyclic graph G with a cycle of length c for which $W(G) = W(G - v)$ if and only if $c \geq 5$. Moreover, we show that every graph G is an induced subgraph of H such that $W(H) = W(H - v)$. As our relaxed version is rich with solutions, it gives hope that Šoltés's problem may have also some solutions distinct from C_{11} .

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1. Introduction

Throughout this paper all graphs will be finite, simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G is usually denoted by $n(G)$. For $u, v \in V(G)$ the distance $d_G(u, v)$ between vertices u and v is defined as the number of edges on a shortest path connecting these vertices in G . The distance, or transmission, $t_G(v)$ of a vertex $v \in V(G)$ is the sum of distances between v and all other vertices of G . By $G - v$ we denote a graph obtained from G when v and all edges incident with v are deleted.

The Wiener index $W(G)$ of a connected graph G is a graph invariant, i.e. a property preserved under all possible isomorphisms of a graph. It is defined as the sum of distances between all (unordered) pairs of vertices in G :

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{v \in V(G)} t_G(v). \quad (1)$$

Wiener index is named after Wiener, who introduced it in 1947. In his article [15] he gave the approximation formula for the boiling point of paraffin which includes the quantity equivalent to the one given by expression (1), realizing that there are correlations between the boiling points of paraffins and the structure of their molecules. Since then, Wiener index has become one of the most frequently used topological indices in chemistry, since molecules are usually modelled by undirected graphs.

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The definition of Wiener index in terms of distances between vertices of a graph, such as in (1), was first given by Hosoya [6]. The same quantity has also been studied in pure mathematics under various names. It seems that the first mathematical paper on Wiener index was published in 1976 [4]. Since then, a lot of mathematicians have studied this quantity very extensively. A great deal of knowledge on Wiener index is accumulated in survey papers [3,7,16]. Wiener index is also closely related to some centrality measures in complex networks. Nowadays it has been frequently used in sociometry and the theory of social networks [5]. Although many papers have been devoted to Wiener index, there are still a lot of open problems and recent researches concerning this quantity; see for instance [1,2,8,10,11]. Therefore, it is still a very popular subject of study in pure and applied mathematics.

In 1991, Šoltés [12] posed the following problem:

Problem 1. Find all such graphs G that the equality $W(G) = W(G - v)$ holds for all their vertices v .

Till now, only one such graph is known: it is a cycle with 11 vertices. This problem is still unsolved, but there are some computational results on graphs which preserve Wiener index after a particular vertex is removed. More precisely, in [3] there are some examples of graphs G with subtree of a same Wiener index. In addition, there are several unicyclic examples that satisfy the requirements of the above problem. Every unicyclic graph contains exactly one cycle and its number of vertices and the number of edges are equal. There are many results concerning Wiener index of unicyclic graphs. We will mention only few of them: Wiener [15] calculated the largest and the smallest Wiener index among all n -vertex unicyclic graphs. Yu and Feng [17] determined the graphs having the largest and smallest Wiener index among all n -vertex unicyclic graphs of given girth. Tan et al. [13,14] studied Wiener index of unicyclic graphs with given girth, maximum degree, number of pendant vertices and cut-vertices.

Motivated by Šoltés's problem and by some examples presented in [3], in this paper we construct an infinite family of unicyclic graphs which preserve Wiener index after removal of a particular vertex. In fact, we show that there are infinitely many unicyclic graphs with this property even when we fix the length of the cycle. Further, we characterize all n 's such that there is a unicyclic graph G with a vertex v for which $W(G) = W(G - v)$. Finally, we show that for every graph G there are infinitely many graphs H such that G is an induced subgraph of H and $W(H) = W(H - v)$ for some vertex $v \in V(H) \setminus V(G)$. Our contribution shows that the class of graphs, which Wiener index does not change when a particular vertex is removed, is rich. This gives hope that Šoltés's problem may have another solution besides C_{11} .

2. Preliminaries

Let G be a connected graph. By $d_G(v)$ we denote the degree of vertex v . A *pendant vertex* is a vertex of degree one and a *pendant edge* is an edge incident with a pendant vertex. One can easily verify the formulae for Wiener index of the path P_n and cycle C_n . Wiener index of path P_n is

$$W(P_n) = \binom{n+1}{3} \quad (2)$$

and Wiener index of a cycle C_n is

$$W(C_n) = \begin{cases} \frac{n^3}{8} & \text{if } n \text{ is even} \\ \frac{n(n^2-1)}{8} & \text{if } n \text{ is odd.} \end{cases} \quad (3)$$

Proposition 2. Let G be a connected graph and $v \in V(G)$ be a pendant vertex. Let uv be the corresponding pendant edge in G and $G' = G - v$. Then

$$W(G) = W(G') + t_{G'}(u) + n(G').$$

The next statement was proved in [9].

Theorem 3. Let G_u and G_v be two graphs with n_u and n_v vertices, respectively, and let $u \in V(G_u)$, $v \in V(G_v)$.

(a) If G arises from G_u and G_v by connecting u and v by an edge, then

$$W(G) = W(G_u) + W(G_v) + n_u t_{G_v}(v) + n_v t_{G_u}(u) + n_u n_v.$$

(b) If G arises from G_u and G_v by identifying u and v , then

$$W(G) = W(G_u) + W(G_v) + (n_u - 1)t_{G_v}(v) + (n_v - 1)t_{G_u}(u).$$

Our construction of unicyclic graphs G for which $W(G) = W(G - v)$ goes in the following way. Let C_c be a cycle of length c . We denote its vertices consecutively by $v_0, v_1, \dots, v_{c-1}$. We add to C_c a pendant vertex, to obtain a new graph, then we add another pendant vertex (which may be connected to previously added vertex) and so on, until we get a unicyclic graph G with $W(G) = W(G - v_0)$. Then we continue with adding pendant vertices to create infinitely many graphs G with the property

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