Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

On diregular digraphs with degree two and excess two

James Tuite

Department of Mathematics and Statistics, Open University, Walton Hall, Milton Keynes, United Kingdom

ARTICLE INFO

Article history:
Received 28 April 2017
Received in revised form 21 September 2017
Accepted 30 October 2017
Available online xxxx

Keywords:
Degree/diameter problem
Digraphs
Excess
Extremal digraphs

ABSTRACT

An important topic in the design of efficient networks is the construction of $(d, k, +\epsilon)$-digraphs, i.e. k-geodetic digraphs with minimum out-degree $\geq d$ and order $M(d, k) + \epsilon$, where $M(d, k)$ represents the Moore bound for degree d and diameter k and $\epsilon > 0$ is the (small) excess of the digraph. Previous work has shown that there are no $(2, k, +1)$-digraphs for $k \geq 2$. In a separate paper, the present author has shown that any $(2, k, +2)$-digraph must be diregular for $k \geq 2$. In the present work, this analysis is completed by proving the nonexistence of diregular $(2, k, +2)$-digraphs for $k \geq 3$ and classifying diregular $(2, 2, +2)$-digraphs up to isomorphism.

1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order $N(d, k)$ of a digraph G with maximum out-degree d and diameter $\leq k$? A simple inductive argument shows that for $0 \leq l \leq k$ the number of vertices at distance l from a fixed vertex v is bounded above by d^l. Therefore, a natural upper bound for the order of such a digraph is the so-called *Moore bound* $M(d, k) = 1 + d + d^2 + \cdots + d^k$. A digraph that attains this upper bound is called a *Moore digraph*. It is easily seen that a digraph G is Moore if and only if it is out-regular with degree d, has diameter k and is k-geodetic, i.e. for any two vertices u, v there is at most one $\leq k$-path from u to v.

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases $d = 1$ or $k = 1$ (the Moore digraphs are directed $(k+1)$-cycles and complete digraphs K_{d+1} respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree d, diameter $\leq k$ and order $M(d, k) - \delta$ for small $\delta > 0$; this is equivalent to relaxing the k-geodecity requirement in the conditions for a digraph to be Moore. δ is known as the *defect* of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the k-geodecity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed k: what is the smallest possible order of a k-geodetic digraph G with minimum out-degree $\geq d$? A k-geodetic digraph with minimum out-degree $\geq d$ and order $M(d, k) + \epsilon$ is said to be a $(d, k, +\epsilon)$-digraph or to have *excess* ϵ. It was shown in [6] that there are no diregular $(2, k, +1)$-digraphs for $k \geq 2$. In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no $(d, k, +1)$-digraphs for $k = 2, 3, 4$ and sufficiently large d. In a separate paper [7], the present author has shown that $(2, k, +2)$-digraphs must be diregular with degree $d = 2$ for $k \geq 2$. In the present paper, we classify the $(2, 2, +2)$-digraphs up to isomorphism and show that there are no diregular $(2, k, +2)$-digraphs for $k \geq 3$, thereby completing the proof of the nonexistence of digraphs with degree $d = 2$ and excess $\epsilon = 2$ for $k \geq 3$. Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect $\delta = 2$.

DOI of original article: http://dx.doi.org/10.1016/j.dam.2017.06.016.
E-mail address: james.tuite@open.ac.uk.

https://doi.org/10.1016/j.dam.2017.10.034
0166-218X/

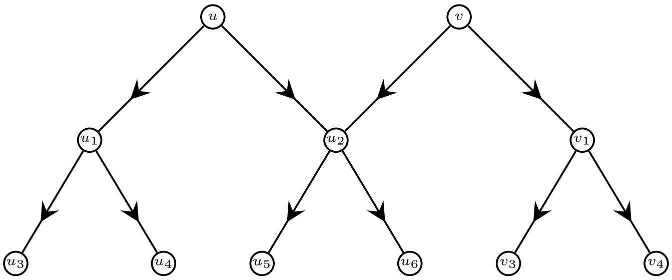

Fig. 1. The vertices u and v.

2. Preliminary results

We will let G stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. G has minimum out-degree $d = 2$, is k-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of G by $V(G)$. By the result of [7], G must be diregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices u and v is the length of the shortest path from u to v. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \to v$ will indicate that there is an arc from u to v. We define the in- and out-neighbourhoods of a vertex u by $N^-(u) = \{v \in V(G) : v \to u\}$ and $N^+(u) = \{v \in V(G) : u \to v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly l from u will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \cup_{i=0}^{l} N^i(u)$ for the set of vertices at distance $\leq l$ from u. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex u of G, there are exactly two distinct vertices that are at distance $\geq k + 1$ from u. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an *outlier set* and its elements *outliers* of u. Notice that $O(u) = V(G) - T_k(u)$. An elementary counting argument shows that in a diregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex u can *reach* a vertex v if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

Lemma 1. *For $k \geq 2$, let u and v be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2)$, $u_2 \in O(u_1)$ and there exists a vertex x such that $O(u) = \{v, x\}$, $O(v) = \{u, x\}$.*

Proof. Suppose that u can reach v by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through v, contradicting k-geodecity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that v cannot reach x by a $\leq k$-path. Similarly, if u_1 can reach u_2 by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \varnothing$, which is impossible. □

Lemma 2. *For $k \geq 2$, there exists a pair of vertices u, v with $|N^+(u) \cap N^+(v)| = 1$.*

Proof. Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \to V(G)$ as follows. Let u^+ be an out-neighbour of a vertex u and let $\phi(u)$ be the in-neighbour of u^+ distinct from u. By our assumption, it is easily verified that ϕ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that G must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

u, v will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_k(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}$, $N^+(u_1) = \{u_3, u_4\}$, $N^+(u_2) = \{u_5, u_6\}$, $N^+(u_3) = \{u_7, u_8\}$, $N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_k(v)$, where we will assume that $u_2 = v_2$.

3. Classification of (2, 2, +2)-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

Theorem 1. *There are exactly two diregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.*

Let G be an arbitrary diregular $(2, 2, +2)$-digraph. G has order $M(2, 2) + 2 = 9$. By Lemma 2, G contains a pair of vertices (u, v) such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of v and v_1 in $T_2(u)$.

Lemma 3. *If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.*

Proof. $v \notin T(u_2)$ by 2-geodecity. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct ≤ 2-paths from u to u_2. Also $v_1 \notin \{u\} \cup T(u_2)$ by 2-geodecity and by assumption $u_1 \neq v_1$. □

Please cite this article in press as: J. Tuite, On diregular digraphs with degree two and excess two, Discrete Applied Mathematics (2017), https://doi.org/10.1016/j.dam.2017.10.034.