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New sufficient conditions for strong unimodality of multivariate discrete distributions

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ABSTRACT

New sufficient conditions that ensure the strong unimodality of multivariate discrete distributions are obtained by the use of a special simplicial subdivision of multidimensional space. Strong unimodality of multivariate Pólya–Eggenberger distribution is shown.

1. Introduction

In the field of optimization, convex analysis plays a crucial role in both theory and practice [3]. While some continuous multivariate functions enjoy a number of useful properties such as convexity, concavity, logconcavity, and other generalized convexity, most of these properties do not directly carry over to the discrete case. In the area of discrete optimization, an analogous theory has been developed and several different types of discrete convexity have been introduced: (i) Miller [9] introduced discretely convex functions. However, the class of discretely convex functions is not closed under addition. (ii) Favati and Tardella [6] introduced integrally convex functions and investigated connections between the convexity of a function on $\mathbb{R}^n$ and the integer convexity of its restriction to $\mathbb{Z}^n$. They presented a polynomial time algorithm to find the minimum of a submodular integrally convex function. Murota and Shioura [14] showed that the class of integrally convex functions is not closed under addition in general. (iii) Murota [10–12] and Murota and Shioura [13–15] have introduced L-convex and M-convex functions and advocated the theory of discrete convex analysis that aims to establish a general theoretical framework for solvable discrete optimization problems by integrating the ideas in continuous optimization and combinatorial optimization. Local minima of the discretely convex functions, integrally convex functions and L/M-convex functions are also global minima, however the definition of locality depends on the type of discrete convexity [23].

Another concept which lies at the very heart of optimization is logconcavity. Logconcavity of continuous multivariate distributions has been extensively studied in literature and a variety of important results has been obtained (see, e.g., [3] and [20]).

A nonnegative function f defined on a convex subset A of the space $\mathbb{R}^n$ is said to be *logconcave* if for every pair $\mathbf{x}, \mathbf{y} \in A$ and $0 < \lambda < 1$, we have the inequality

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq [f(\mathbf{x})]^{\lambda}[f(\mathbf{y})]^{(1-\lambda)}.$$

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If f is positive valued, then $\log f$ is a concave function on A. If the inequality holds strictly for $\mathbf{x} \neq \mathbf{y}$, then f is said to be strictly logconcave.

The notion of logconcave probability measures was introduced by Prékopa [18,19]. A probability measure P, defined on $(\mathbb{R}^n, \mathscr{B}_n)$, is said to be logconcave if for every pair of nonempty convex sets $A, B \subset \mathbb{R}^n$ (any convex set is Borel measurable), we have the inequality

$$P(\lambda A + (1-\lambda)B) \geq [P(A)]^{\lambda}[P(B)]^{(1-\lambda)},$$

where the $+$ sign refers to Minkowski addition of sets, i.e.,

$$\lambda A + (1-\lambda)B = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \mid \mathbf{x} \in A, \mathbf{y} \in B\}.$$

The above notion generalizes in a natural way to nonnegative valued measures. In this case we require the logconcavity inequality to hold for finite $P(A), P(B)$. The notion of logconcave probability measures has revolutionized the theory of logconcavity and led to a wide range of applications in stochastic programming (see, e.g., [5,4,16,18–20]), economics (see, e.g., [1]), probability and statistics (see, e.g., [8,24,25]) just to name a few.

While the logconcavity of probability density functions has attracted considerable attention, little attention has been given to the logconcavity of discrete distributions. The classical result in this respect is due to Fekete (1912) who introduced the notion of an *r-times positive sequence*. The sequence of nonnegative elements $\ldots, a_{-2}, a_{-1}, a_0, \ldots$ is said to be *r-times positive* if the matrix

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ \ldots & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & \ldots \\ \ldots & a_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

has no negative minor of order smaller than or equal to r.

Twice-positive sequences are those for which we have

$$\begin{vmatrix} a_i & a_j \\ a_{i-t} & a_{j-t} \end{vmatrix} = a_i a_{j-t} - a_j a_{i-t} \geq 0 \tag{1.1}$$

for every $i < j$ and $t \geq 1$. This holds if and only if $a_i^2 \geq a_{i-1}a_{i+1}$. Fekete [7] also proved that the convolution of two r-times positive sequences is r-times positive. Twice-positive sequences are also called *logconcave sequences*. Hence, Fekete's theorem states that the convolution of two logconcave sequences is logconcave. A discrete probability distribution, defined on the real line, is said to be logconcave if the corresponding probability mass function (p.m.f) is logconcave.

Following Barndorff-Nielsen [2] a joint p.m.f p of a random vector $X \in \mathbb{Z}^n$ is called *strongly unimodal* if there exists a convex function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ such that

$$f(\mathbf{x}) = -\log p(\mathbf{x}) \text{ if } \mathbf{x} \in \mathbb{Z}^n.$$

If $p(\mathbf{x}) = 0$, then $f(\mathbf{x}) = \infty$ by definition. This notion is not a direct generalization of that of the one-dimensional case, i.e., of formula (1.1). However, the two notions are the same when $n = 1$ (see, e.g., [20]). Note that if p is strongly unimodal, then it is logconcave. In general, the convolution of two logconcave p.m.f's defined on $\mathbb{Z}^n$ is no longer logconcave if $n \geq 2$.

Pedersen [17] used the strong unimodality definition of Barndorff-Nielsen and presented two sufficient conditions for a bivariate discrete distribution to be strongly unimodal. Pedersen [17] also proved that the trinomial distribution is logconcave and the convolution of any finite number of these distributions with possibly different parameter sets is also logconcave.

Subasi et al. [22] presented six sufficient conditions for a trivariate discrete distribution to be strongly unimodal and one sufficient condition for multivariate discrete distributions defined on $\mathbb{Z}^n$. Subasi et al. [22] also showed that negative multinomial, multivariate hypergeometric, multivariate negative hypergeometric, and Dirichlet (or beta)-compound multinomial distributions are strongly unimodal.

The goal of this paper is to extend the results of Subasi et al. [22] by providing new sufficient conditions that ensure the strong unimodality of multivariate discrete distributions. The organization of the paper is as follows. In Section 2 we present all possible subdivisions of a cube into simplices with disjoint interiors and use them to obtain all possible sufficient conditions for a trivariate discrete distribution to be strongly unimodal. In Section 3 we present a new sufficient condition that ensures the strong unimodality of multivariate discrete distributions. Finally, in Section 4, we present a multivariate discrete distribution whose joint p.m.f satisfies the new sufficient condition for strong unimodality.

2. Sufficient conditions for strong unimodality of trivariate discrete distributions

In this section we shall present sufficient conditions for a discrete distribution defined on $\mathbb{Z}^3$ to be strongly unimodal. In order to fully describe the sufficient conditions that ensure strong unimodality of trivariate distributions we first investigate all possible subdivisions of a cube into simplices (tetrahedra) with disjoint interiors. We shall call two tetrahedra "neighbors" if they share a common face.

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