Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On facial unique-maximum (edge-)coloring

Vesna Andova [a,b], Bernard Lidický [c], Borut Lužar [b,*], Riste Škrekovski [b,d,e]

[a] *Faculty of Electrical Engineering and Information Technologies, Ss Cyril and Methodius University, Skopje, Macedonia*
[b] *Faculty of Information Studies, Novo mesto, Slovenia*
[c] *Department of Mathematics, Iowa State University, USA*
[d] *University of Ljubljana, Faculty of Mathematics and Physics, Slovenia*
[e] *University of Primorska, FAMNIT, Koper, Slovenia*

ARTICLE INFO

Article history:
Received 19 July 2017
Accepted 20 November 2017
Available online xxxx

Keywords:
Facial unique-maximum coloring
Facial unique-maximum edge-coloring
Plane graph

ABSTRACT

A *facial unique-maximum coloring* of a plane graph is a vertex coloring where on each face α the maximal color appears exactly once on the vertices of α. If the coloring is required to be proper, then the upper bound for the minimal number of colors required for such a coloring is set to 5. Fabrici and Göring (2016) even conjectured that 4 colors always suffice. Confirming the conjecture would hence give a considerable strengthening of the Four Color Theorem. In this paper, we prove that the conjecture holds for subcubic plane graphs, outerplane graphs and plane quadrangulations. Additionally, we consider the facial edge-coloring analogue of the aforementioned coloring and prove that every 2-connected plane graph admits such a coloring with at most 4 colors.

1. Introduction

In this paper, we consider simple graphs only. We call a graph *planar* if it can be embedded in the plane without crossing edges and we call it *plane* if it is already embedded in this way. A *coloring* of a graph is an assignment of colors to vertices. If in a coloring adjacent vertices receive distinct colors, it is *proper*. The cornerstone of graph colorings is the Four Color Theorem stating that every planar graph can be properly colored using at most 4 colors [1]. Fabrici and Göring [5] proposed the following strengthening of the Four Color Theorem.

Conjecture 1 (*Fabrici and Göring [5]*)**.** *If G is a plane graph, then there is a proper coloring of the vertices of G by colors in* $\{1, 2, 3, 4\}$ *such that every face contains a unique vertex colored with the maximal color appearing on that face.*

A proper coloring of a graph embedded on some surface, where colors are integers and every face has a unique vertex colored with a maximal color, is called a *facial unique-maximum coloring* or *FUM-coloring* for short (Wendland uses the notion *capital coloring* instead). This type of coloring was first studied by Fabrici and Göring [5]. The main motivation for their research comes from the *unique-maximum coloring* (also known as *ordered coloring*), defined as a coloring where there is only one vertex colored with the maximal color on every path in a graph. Studying unique-maximum coloring was motivated due to a number of applications it finds in various branches of mathematics and computer science; see, e.g., [2,3,7] for more details. Fabrici and Göring used this concept in a facial version, which is of great interest, among others, also due to Conjecture 1 and its direct connection to the Four Color Theorem. Coloring embedded graphs with respect to faces is a bursting field itself; the main directions are presented in a recent survey by Czap and Jendrol' [4].

* Corresponding author.
E-mail addresses: vesna.andova@gmail.com (V. Andova), lidicky@iastate.edu (B. Lidický), borut.luzar@gmail.com (B. Lužar), skrekovski@gmail.com (R. Škrekovski).

https://doi.org/10.1016/j.dam.2017.11.024
0166-218X/

For a graph G, the minimum number k such that G admits a FUM-coloring with colors $\{1, 2, \dots, k\}$ is called the *FUM chromatic number of* G, denoted by $\chi_{\rm fum}(G)$. Fabrici and Göring [5] proved that if G is a plane graph, then $\chi_{\rm fum}(G) \leq 6$. Their result was further improved as follows.

Theorem 1 (*Wendland [9]*)**.** *If G is a plane graph, then $\chi_{\rm fum}(G) \leq 5$.*

We show that the upper bound 4 from Conjecture 1 holds for several subclasses of plane graphs, and that, surprisingly, the bound is tight in most of the cases. The main result of the paper regarding the FUM-coloring of vertices is the following.

Theorem 2. *If G is a plane subcubic graph or an outerplane graph, then $\chi_{\rm fum}(G) \leq 4$.*

In the second part of the paper, we consider the edge-coloring version of the problem, which has been introduced by Fabrici, Jendrol', and Vrbjarová [6]. For a graph G embedded on some surface, two distinct edges are said to be *facially adjacent* if they are consecutive in some facial path, i.e., they have a common vertex and they are incident with a same face. A *facial edge-coloring* is a coloring of edges such that facially adjacent edges receive distinct colors. It is rather straightforward to prove that every plane graph admits a facial edge-coloring with at most 4 colors.

For a graph G, we denote by $\chi'_{\rm fum}(G)$ the minimum number k such that there exists a facial edge-coloring using colors $1, \dots, k$ such that each face is incident with a unique edge colored with the maximal color. Such a coloring is called a *FUM-edge-coloring*. In [6], Fabrici et al. proposed the following conjecture.

Conjecture 2 (*Fabrici et al. [6]*)**.** *If G is a 2-edge-connected plane graph, then $\chi'_{\rm fum}(G) \leq 4$.*

In [6], the authors proved that $\chi'_{\rm fum}(G) \leq 6$ for every 2-edge-connected plane graph G. Our main result is that we prove $\chi'_{\rm fum}(G) \leq 4$ if the assumption that the graph is 2-edge-connected is replaced by 2-vertex-connectivity, supporting Conjecture 2.

Theorem 3. *If G is a 2-vertex-connected plane graph, then $\chi'_{\rm fum}(G) \leq 4$.*

Observe that every edge in an embedded graph is facially adjacent to at most four other edges, therefore one can translate the problem of facial edge-coloring of a plane graph to a vertex coloring of a plane graph with maximum degree 4. Hence, Theorem 1 directly implies $\chi'_{\rm fum}(G) \leq 5$ for every plane graph G. Similarly, Theorem 2 implies that if G is obtained from a plane graph by subdividing every edge, then $\chi'_{\rm fum}(G) \leq 4$.

The paper is organized as follows. In Section 2, we prove Theorem 2 and discuss the FUM-coloring of vertices. In Section 3, we consider the FUM-edge-coloring and prove Theorem 3. Both proofs, of Theorems 2 and 3, use precoloring extension technique successfully applied by Thomassen [8] when proving that every planar graph is 5-choosable. In Concluding remarks, we present some related results and discuss possible future directions on this topic.

2. FUM-(vertex-)coloring

In this section we consider the FUM-coloring of vertices and confirm that Conjecture 1 holds for several subclasses of plane graphs.

First, we recall a theorem, which is the main tool used in [5], and will prove helpful also in proving our results.

Theorem 4 (*Fabrici and Göring [5]*)**.** *Every plane graph has a (not necessarily proper) 3-coloring with colors black, blue, and red such that*

(1) *each face is incident with at most one red vertex,*
(2) *each face that is not incident with a red vertex is incident with exactly one blue vertex.*

A slightly stronger version of Theorem 4 was proved by Wendland [9] who also added the conclusion that each triangle, facial or separating, contains at least one vertex that is not black. This enabled him to improve the upper bound to 5 colors.

Recall that Conjecture 1 states that if G is a plane graph, then its FUM chromatic number is 4, which is the same upper bound as for the chromatic number. One can therefore ask, which are the plane graphs admitting a FUM-coloring with at most 3 colors. However, natural candidates such as graphs of large girth, quadrangulations, and outerplane graphs have infinitely many examples with FUM chromatic number 4.

The example in Fig. 1 shows that there is no analogue of Grötzsch's result for the FUM-coloring. Indeed, every vertex lies on the outer face, and hence only one can be colored with 3 (assuming 3 colors suffice). As every vertex is incident to at most three faces, the maximal color of the fourth face is 2, and hence all the other vertices should receive 1, which is not possible, since the coloring must be proper.

We continue by considering plane quadrangulations.

Proposition 1. *If G is a plane quadrangulation, then $\chi_{\rm fum}(G) \leq 4$. Moreover, there exists an infinite family of plane quadrangulations with FUM chromatic number 4.*