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The convexity of induced paths of order three and applications: Complexity aspects

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ABSTRACT

In this paper, we introduce a new convexity on graphs similar to the well known P_3-convexity, which we will call P_3^*-convexity. We show that several P_3^*-convexity parameters (hull number, convexity number, Carathéodory number, Radon number, interval number and percolation time) are NP-hard even on bipartite graphs. We prove a strong relationship between this convexity and the well known geodesic convexity, which implies several NP-hardness results for the latter. In order to show that, we prove that the hull number for the P_3-convexity is NP-hard even for subgraphs of grids and that the convexity number for the P_3-convexity is NP-hard even for bipartite graphs with diameter 3. We also obtain linear time algorithms to determine those parameters for the above mentioned convexities for cographs and P_4-sparse graphs.

1. Introduction

Convexity spaces have been considered in different branches of mathematics. The study of convexities applied to graphs has started more recently, about 50 years ago [14,19,20]. Abstract convexity parameters, when considered on graph convexities, give rise to interesting graph parameters. In particular, complexity aspects related to the computation of these parameters are the main goal of various recent papers.

The computation of convexity parameters for a graph depends on the particular convexity being considered. Among the existing convexities we can mention the following, whose convex sets are based on paths of the graph: geodesic, monophonic, P_3, m^3 and triangle-path convexities. They are defined by letting the convex sets be closed, respectively, under shortest paths [8], induced paths [10], paths of order 3 [5], induced paths of length at least 3 [13] and T-paths [6] (paths which allow only chords between vertices at distance 2 in the path).

In this paper, we introduce the P_3^*-convexity, where the convex sets are closed under induced paths of length 2. That is, a subset $S \subseteq V(G)$ is P_3^*-convex if every vertex in an induced path of length 2 between two vertices in S also belongs to S. Equivalently, a set S is P_3^*-convex if no vertex outside S has two non-adjacent neighbors in S.

A motivation for studying the proposed convexity is that it fills an existing gap considering the monophonic and the m^3-convexities, while being close to the P_3-convexity. At the same time, it turns out that there is also a strong relationship between the proposed P_3^*-convexity and the geodesic convexity, since every geodesic convex set is P_3^*-convex.

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This relationship automatically implies several NP-hardness results on parameters for the geodesic convexity, providing a common framework for these proofs, as we show in Section 5. As for the P_3^*-convexity, itself, in Section 3 we show that the mentioned parameters are all NP-hard. In fact, we show that the convexity number for the P_3-convexity is NP-hard even for bipartite graphs with diameter 3 and that the hull number for the P_3-convexity is NP-hard even for subgraphs of grids. Moreover, we obtain linear time algorithms for some graph classes in Section 6. In the next section, we give some useful definitions.

2. Definitions

Let G be a simple finite graph, with vertex set $V(G)$ and $\mathcal{C}$ a family of subsets of $V(G)$. The pair $(G, \mathcal{C})$ is a *graph convexity*, when $\emptyset \in \mathcal{C}$, $V(G) \in \mathcal{C}$ and, if $S_1, S_2 \in \mathcal{C}$, then $S_1 \cap S_2 \in \mathcal{C}$. The subsets $C \in \mathcal{C}$ are called *convex sets*. We say that a set is *co-convex* if its complement is convex. The *convex hull* of a subset $S \subseteq V(G)$ with respect of a graph G and a convexity $\mathcal{C}$, denoted by $H_{\mathcal{C},G}(S)$, is the smallest convex set which contains S. When the convexity and graph being considered are clear from the context, we will omit the subscript. If $H(S) = V(G)$, we say that S is a *hull set*.

Given a subset $S \subseteq V(G)$, let $I(S)$ be the set with the vertices in S and all vertices in a certain type of path between two vertices of S and let $I^k(\cdot)$ be the kth iterate of $I(\cdot)$, i.e., $I^0(S) = S$ and $I^k(S) = I(I^{k-1}(S))$ for $k \geq 1$. The kind of path considered for $I(\cdot)$ is defined by the convexity being considered. In this paper we are concerned with paths of order 3, corresponding to the P_3-convexity, induced paths of order 3, corresponding to the P_3^*-convexity, and shortest paths, corresponding to the geodesic convexity or simply g-convexity. If $I(S) = V(G)$, we say that S is an *interval set*.

Let $\mathcal{C}$ be a graph convexity. Next, we describe some graph parameters relative to the $\mathcal{C}$-convexity. The *hull number* $h_{\mathcal{C}}(G)$ of G is the size of a minimum hull set. The *interval number* $n_{\mathcal{C}}(G)$ is the size of a minimum interval set. The *convexity number* $cx_{\mathcal{C}}(G)$ is the size of the maximum convex set distinct from $V(G)$. The *Radon number* $r_{\mathcal{C}}(G)$ is the minimum k such that every subset V' of $V(G)$ of size at least k has a partition (V'_1, V'_2) such that $H(V'_1) \cap H(V'_2) \neq \emptyset$.

The *percolation time* $t_{\mathcal{C},G}(S)$ of a set S of vertices is the minimum k such that $I^k(S) = I^{k+1}(S)$. The *percolation time* $t_{\mathcal{C}}(G)$ is the maximum percolation time $t_{\mathcal{C},G}(S)$ among all hull sets S of G.

The *Carathéodory number* $c_{\mathcal{C}}(G)$ is the smallest integer c such that for every set S and every vertex $u \in H(S)$, there is a set $F \subseteq S$ with $|F| \leq c$ and $u \in H(F)$. Alternatively, the Carathéodory number $c_{\mathcal{C}}(G)$ is the size of the maximum *Carathéodory set*, where we say that a set $S \subseteq V(G)$ is a Carathéodory set if $\partial H(S) = H(S) \setminus \bigcup_{s \in S} H(S \setminus \{s\}) \neq \emptyset$.

Throughout the text, when the convexity and the graph being considered are clear from context we will omit the subscripts. Otherwise, we will use the subscripts P_3, P_3^* or g (for the g-convexity). For example, if $S \subseteq V(G_1) \subseteq V(G_2)$, then $H_{g,G_1}(S)$ and $t_{P_3,G_2}(S)$ are respectively the g-convex hull of S in G_1 and the P_3 percolation time of S in G_2.

3. NP-hardness results for the hull number

In this section we establish NP-hardness results for various convexity parameters related to the P_3- and P_3^*- convexities. Observe that, in triangle-free graphs, the P_3^*-convexity is identical to the P_3-convexity, since every path of length 2 is induced. Since from [2,18,5,12] the P_3-Carathéodory number, the P_3-Radon number, the P_3-percolation time and the P_3-interval number are NP-hard on bipartite graphs, we have directly that:

Theorem 1. *The problems of finding the Carathéodory number, the Radon number, the interval number and the percolation time are NP-hard on bipartite graphs for the P_3^*-convexity.*

In 2011, Centeno et al. [4] proved that the problem of finding the hull number is NP-hard for the P_3-convexity. We extend this result for planar bipartite graphs with maximum degree $\Delta \leq 4$.

Theorem 2. *The problem of finding the hull number is NP-hard even on planar bipartite graphs with maximum degree $\Delta \leq 4$ for the P_3- and P_3^*- convexities.*

Proof. We obtain a reduction from Planar 3- SAT with restrictions. Given a set $\mathcal{C} = \{C_1, \ldots, C_m\}$ of clauses as an instance of SAT, the *underlying graph of* $\mathcal{C}$ is a graph which has one vertex for each literal (which is a variable or the complement of a variable), has one vertex for each clause, has an edge between each literal vertex and its corresponding complement and has an edge between a literal vertex and a clause vertex if and only if the corresponding clause contains the corresponding literal. If the underlying graph of $\mathcal{C}$ is planar, we say that $\mathcal{C}$ is a *planar formula*. We say that $\mathcal{C}$ is a *restricted formula* if each clause has at most three literals, each variable appears in at most three clauses and every literal of the form $\bar{x}$, where x is a variable, appears in exactly one clause. We prove a reduction from Planar 3- SAT, which is known to be NP-complete [17], even for restricted planar formulae (see the proof of Theorem 2a in [7]).

Restricted Planar 3- SAT
Input: A SAT restricted planar formula $\mathcal{C}$ on variables of a set X.
Question: Is there a truth assignment to X that satisfies all clauses of $\mathcal{C}$?

Given a restricted planar formula $\mathcal{C}$ with k variables and m clauses, we construct a bipartite graph G such that $\mathcal{C}$ is satisfiable if and only if the P_3-hull number of G is $12k + m$. The variable gadget related to any variable is shown in Fig. 1. The clause