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Infection in hypergraphs

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ABSTRACT

In this paper a new parameter for hypergraphs called *hypergraph infection* is defined. This concept generalizes zero forcing in graphs to hypergraphs. The exact value of the infection number of complete and complete bipartite hypergraphs is determined. A formula for the infection number for interval hypergraphs and several families of cyclic hypergraphs is given. The value of the infection number for a hypergraph whose edges form a symmetric t-design is given, and bounds are determined for a hypergraph whose edges are a t-design. Finally, the infection numbers for several hypergraph products and line graphs are considered.

1. Introduction

The subject of zero forcing for graphs has been widely studied [3,4,1,11]. In this paper we generalize the concept of zero forcing on graphs to hypergraphs.

In standard zero forcing for graphs, the vertices of the graph are coloured either black or white. A black vertex can force a white vertex to black according to a colour change rule. The colour change rule for standard zero forcing is that a black vertex can force an adjacent white vertex to black if it is the only white vertex adjacent to that black vertex. A set of vertices in a graph is a *zero forcing set* for the graph if, when the vertices in this set are set to black and the colour change rule is applied repeatedly, all the vertices of the graph are eventually forced to black. The *zero forcing number* of a graph is the size of the smallest zero forcing set for the graph. For a graph G, the zero forcing number is denoted by $Z(G)$.

The term "zero forcing" is based on an algebraic property of these sets. Consider a vector with the entries corresponding to the vertices of a graph. Further, assume the entries corresponding to a set of vertices in a zero forcing set for the graph are equal to zero. The zero forcing property of the set guarantees that such a vector is in the kernel of the adjacency matrix of the graph only if the vector is the zero vector. The term "zero forcing" refers to the fact that the remaining entries of the vector are forced to be zero for the vector to be in the kernel of the adjacency matrix.

For hypergraphs, there is no matrix analogous to the adjacency matrix of a graph and this notion of a set of entries in a vector forcing the other entries to be zero in a proposed null vector does not apply. However, in this paper we focus on generalizing the colour change rule and hence we use the term *infection* rather than zero forcing. Terms such as infection, propagation, k-forcing, probabilistic zero forcing, iteration, and searching have all been used for notions similar to zero forcing; see, for example, [10,14,15,2,12,6].

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For infection in a hypergraph, the vertices are initially either infected or uninfected (as opposed to being coloured either black or white as they are in zero forcing for graphs). There is an *infection rule* that determines when vertices can infect other vertices which is analogous to the zero forcing rule for graphs. In the case of hypergraphs, it is a subset of infected vertices in an edge that may infect the remaining vertices in that specific edge, rather than a single vertex forcing another vertex. The following is the infection rule for hypergraphs.

Infection Rule: A non-empty set A of infected vertices can infect the vertices in an edge E if:

1. $A \subset E$, and
2. if v is an uninfected vertex such that $A \cup \{v\}$ is a subset of some edge in the hypergraph, then $v \in E$.

Similar to the case for graphs, if two vertices in a hypergraph are contained in a common edge, then we say that the vertices are *adjacent*. Further, for a hypergraph, we can define two sets to be adjacent if there is an edge that contains them both. So a set A of infected vertices can infect an edge E if there are no uninfected vertices outside of E that are adjacent to A.

If $A \subset E$ satisfies the conditions in the infection rule, then we say that "the set A infects the edge E". In the case of hypergraphs, it is an edge and all the vertices in the edge that are infected rather than a single vertex, as is the case for graphs. A set of vertices in a hypergraph is an *infection set* if, when the vertices in the set are initially infected and the infection rule is applied repeatedly, all the vertices in the hypergraph become infected. The *infection number* of a hypergraph H is the size of a smallest infection set for H; the infection number of H is denoted by $I(H)$.

In a hypergraph, the edges are subsets of the vertices and can be of any size. (The *size* of an edge is the number of vertices in the edge.) If all the edges in a hypergraph contain exactly k vertices, then the hypergraph is called a *k-hypergraph*. A 2-hypergraph is a graph; the infection number for a 2-hypergraph is equivalent to the zero forcing number of the graph.

Proposition 1.1. *Let H be a 2-hypergraph. Then $Z(H) = I(H)$.*

Proof. In a 2-hypergraph, the condition that a vertex v can apply a force is equivalent to the condition that a proper subset of vertices (which in this case is a singleton) can infect an edge. □

Let H be a hypergraph and W a subset of the vertices of H. The set of all vertices in H that are infected after repeatedly applying the infection rule, with W being the set of initially infected vertices, is called the *derived set of* W. This is denoted by I_W. A set W is an infection set if and only if the derived set is the set of all vertices.

The empty hypergraph is the hypergraph with no vertices and no edges; we will not consider this case. A *trivial hypergraph* is a hypergraph with vertices but no edges. The infection number for any trivial hypergraph is clearly the number of vertices in the hypergraph since no set can ever infect any edge. For every other hypergraph, there is an upper bound on the size of the infection number for a hypergraph.

Proposition 1.2. *Let H be a non-trivial hypergraph on n vertices and let k be the size of the largest edge in H. Then*

$$I(H) \leq n - k + 1.$$

Proof. Let A be a $(k-1)$-subset of an edge E of size k in the hypergraph. We claim that the set of all vertices not in A forms an infection set of size $n-(k-1)$ for the hypergraph. This follows since the final element in E cannot be adjacent to any uninfected vertices outside of E, as there are no uninfected vertices outside of A, and thus can infect E. □

In Section 3, we will see that this bound holds with equality for the complete hypergraph. We will also demonstrate other hypergraphs where this bound is tight.

The *line graph* of a hypergraph H is the graph formed by representing each edge of H by a vertex. These vertices are adjacent in the line graph if and only if the corresponding edges of the hypergraph H intersect. The line graph of a hypergraph H is denoted by $L(H)$; we consider these graphs in more detail in Section 7. A hypergraph is *connected* if and only if its line graph is connected, that is, a hypergraph is connected if there is a path between any two vertices in the line graph. A *connected component* of a hypergraph is a maximal connected sub-hypergraph. It is not hard to see that the infection number of a hypergraph is the sum of the infection numbers of the connected components of the hypergraph.

Proposition 1.3. *If H is a hypergraph with connected components $H_1, \ldots, H_t$ then*

$$I(H) = \sum_{i=1}^{t} I(H_i).$$

2. Reduced hypergraphs

One significant difference between graphs and hypergraphs is that there is no restriction on the size of an edge in a hypergraph, while in a graph all edges have size two. This can lead to the situation where one edge of a hypergraph is a subset of another edge. The most extreme case of this is when the entire set of vertices forms an edge.