Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Singularities in Negami's splitting formula for the Tutte polynomial

J.M. Burgos

Center for Research and Advanced Studies of the National Polytechnic Institute, Mathematics Department, CINVESTAV, Mexico City, 07360, Mexico

ARTICLE INFO

Article history:
Received 5 July 2016
Received in revised form 11 November 2017
Accepted 17 November 2017
Available online xxxx

Keywords:
Tutte polynomial
Graph Theory
Splitting formula

ABSTRACT

The n-sum graph Negami's splitting formula for the Tutte polynomial is not valid in the region $(x-1)(y-1)=q$ for $q=1,2,\ldots n-1$ with the additional region $y=1$ if $n>3$. This region corresponds to (up to prefactors and change of variables) the Ising model, the q-state Potts model, the number of spanning forest generator and particularizations of these. We show splitting formulas for these specializations.

1. Introduction

The Tutte polynomial of a graph G, also known as dichromate or Tutte–Whitney polynomial, is defined as the following subgraph generating function [21]:

$$T(G;x,y)=\sum_{\substack{A\subseteq G\\ V(A)=V(G)}}(x-1)^{\omega(A)-\omega(G)}\,(y-1)^{\omega(A)+|E(A)|-|V(G)|}$$

where $A\subseteq G$ indicates that A is a subgraph of G and $\omega(G)$ denotes the number of connected components of G. It is the most general graph invariant that can be defined by the *deletion–contraction algorithm*:

$$T(G;x,y)=T(G/e;x,y)+T(G-e,x,y)$$

where e is neither a loop (and edge with coincident endpoints) nor a bridge (an edge whose deletion increases the number of connected components), with $T(G;x,y)=x^iy^j$ if the edge set of G only has i bridges and j loops. Here G/e and $G-e$ denote the contraction and deletion of the edge e respectively. Computing the Tutte polynomial is in general an NP-hard problem [12].

Different specializations with respective prefactors and change of variables of the Tutte polynomial, naturally appear as classical invariants in several branches of mathematics, physics and engineering [1,4,3,7]. For example, the Jones polynomial in knot theory [13], the reliability polynomial in network engineering, the Ising and Potts model in statistical mechanics [11,18,19], the random cluster model [10], etc. (see Table 1).

Following [16], assume that the graph G splits in subgraphs K and H only sharing n common vertices $U=V(K)\cap V(H)$. Let $\Gamma(U)$ denote the partition lattice over U and let $\mathcal{A}=\{U_1,U_2,\ldots,U_k\}$ be one of these partitions. Denote by $K/\mathcal{A}$ and $H/\mathcal{A}$ the graphs obtained by identifying all vertices in each U_i of K and H respectively, see Fig. 1. The following is Negami's splitting formula for the Tutte polynomial (Corollary 4.7, iv, [16]):

$$T(G;x,y)=\sum_{\mathcal{A},\mathcal{B}\in\Gamma(U)}c_{\mathcal{A}\mathcal{B}}(x,y)\,T(K/\mathcal{A};x,y)\,T(H/\mathcal{B};x,y)\qquad(1)$$

E-mail address: burgos@math.cinvestav.mx.

https://doi.org/10.1016/j.dam.2017.11.021

Table 1
Specializations of the Tutte polynomial up to prefactors and change of variables.

Specialization	Invariant
$xy = 1$	Jones polynomial
$y = 0$	Chromatic polynomial
$x = 1,\ y \neq 1$	Reliability polynomial
$x = 0$	Flow polynomial
$(x-1)(y-1) = 2$	Ising model
$(x-1)(y-1) = q$	q-state Potts model
$y \neq 1$	Random cluster model
$y = 1$	Number of spanning forest generator
$(1, 1)$	Number of spanning tree
$(2, 1)$	Number of spanning forest
$(1, 2)$	Number of spanning subgraph

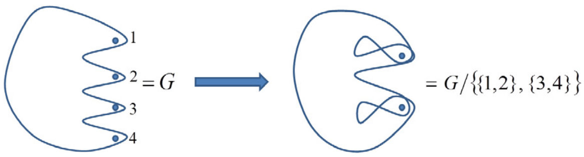

Fig. 1. Identification of vertices.

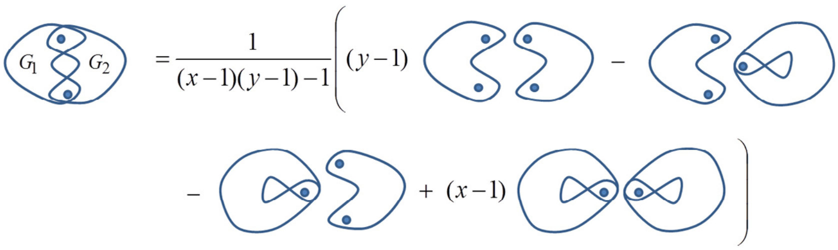

Fig. 2. Two sum Tutte polynomial splitting.

where $c_{\mathcal{AB}}(x, y)$ are rational functions of x and y on the field of rational numbers. A colored version of this formula was developed in [20] and the case of Tutte polynomials of generalized parallel connections of general matroids can be found in [5].[1] Explicit splitting formulas were also given in [17] and [2]. As an application of the Feferman–Vaught Theorem, the existence of splitting formulas for a wide class of graph polynomials which includes the Tutte polynomial is proved in [15]. The above result is an existential theorem, it is not explicit like the others.

In view that some denominators of the coefficients $c_{\mathcal{AB}}$ could annihilate restricted to certain regions, we wonder whether the formula holds for different specializations. For example, for the 2-sum such that H and K are connected we have the Brylawski sum ([6], Corollary 6.14)[2] :

$$T(G; x, y) = \frac{1}{(x-1)(y-1)-1}\Big((y-1)\, T(K; x, y)\, T(H; x, y) - T(K; x, y)\, T(H/\mathcal{A}; x, y) - T(K/\mathcal{A}; x, y)\, T(H; x, y) + (x-1)\, T(K/\mathcal{A}; x, y)\, T(H/\mathcal{A}; x, y)\Big) \qquad (2)$$

where $\mathcal{A}$ is the trivial or minimal partition of the common two vertices between H and K. Fig. 2 shows this factorization. Is clear that this formula does not hold in the region $(x-1)(y-1) = 1$.

In the next section we prove the following: For the n-sum graph, Negami's formula (1) holds only over the region $(x-1)(y-1) \neq q$ such that $q = 1, 2 \ldots n-1$ with the additional constraint $y \neq 1$ if $n > 3$. The region where Negami's formula does not hold will be called the *singular region*.

[1] The author is grateful to Prof. Lorenzo Traldi and Prof. Anna de Mier for these references and valuable comments.

[2] As far as the author knows, this was the second known splitting formula for the Tutte polynomial after the well known factorization through an articulation point.

Please cite this article in press as: J.M. Burgos, Singularities in Negami's splitting formula for the Tutte polynomial, Discrete Applied Mathematics (2017), https://doi.org/10.1016/j.dam.2017.11.021.