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The matcher game played in graphs

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ABSTRACT

We study a game played on a graph by two players, named Maximizer and Minimizer. Each round two new vertices are chosen; first Maximizer chooses a vertex u that has at least one unchosen neighbor and then Minimizer chooses a neighbor of u. This process eventually produces a maximal matching of the graph. Maximizer tries to maximize the number of edges chosen, while Minimizer tries to minimize it. The matcher number $\alpha'_g(G)$ of a graph G is the number of edges chosen when both players play optimally. In this paper it is proved that $\alpha'_g(G) \geq \frac{2}{3}\alpha'(G)$, where $\alpha'(G)$ denotes the matching number of graph G, and this bound is tight. Further, if G is bipartite, then $\alpha'_g(G) = \alpha'(G)$. We also provide some results on graphs of large odd girth and on dense graphs.

1. Introduction

In this paper we introduce the *matcher game*. This is played on a graph G by two players, called *Maximizer* and *Minimizer*, who take turns in constructing a matching of G. Each round, one player chooses a vertex u_i with at least one neighbor not previously chosen, and the other player chooses a vertex v_i not previously chosen that is a neighbor of u_i. This process continues until no more play is possible; that is, the edges $u_i v_i$ form a maximal matching of G. Maximizer wishes to maximize the number of edges in this matching, while Minimizer wishes to minimize it. The *Max-start matcher number* $\alpha'_g(G)$ of G is the number of edges in the matching when Maximizer starts and both players play optimally, while the *Min-start matcher number* $\hat{\alpha}'_g(G)$ of G is the number of edges chosen when Minimizer starts and both players play optimally.

The matcher game is related to the growing family of *competition parameters* or *competitive optimization games* on graphs and hypergraphs. For example, Phillips and Slater [12] introduced a game where two players alternate add vertices of G to an independent set until it becomes a maximal independent set. One player wants the final set to be large; the other wants it to be small. More recently, Cranston et al. [6] introduced a game where the players alternate adding edges of G to a matching until it becomes a maximal matching. Theirs is equivalent to the game of [12] played on the line graph and is a very different game to ours. Probably the best-known such parameter is the *game chromatic number*, which was introduced by Brams for planar graphs (cf. [8]) and independently by Bodlaender [1] for general graphs, and for list colorings by Borowiecki et al. [2]. There is also work on domination [3,9,10], transversals in hypergraphs [4,5] and more.

We denote the matching number of graph G by $\alpha'(G)$, and define the *lower matching number* $\alpha'_L(G)$ as the minimum cardinality of a maximal matching of G. Trivially the values of our game are sandwiched between $\alpha'_L(G)$ and $\alpha'(G)$. If these two values are equal – that is, the graph is *equimatchable* – then the value of the game is determined. But in general, for the case that Maximizer starts, the value can lie between these two bounds.

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In this paper, we will observe that the Min-start matcher number is uninteresting, as it always equals the ordinary matching number. Thus the focus is on the Max-start matcher number which we simply call the *matcher number*. There we show that, while for bipartite graphs the matcher number equals the matching number, in general $\alpha'_g(G) \geq \frac{2}{3}\alpha'(G)$, and we characterize the graphs G for which $\alpha'_g(G) = \frac{2}{3}\alpha'(G)$. We also investigate the relationship with the lower matching number, showing that $\alpha'_g(G) \geq 2\alpha'(G) - 2\alpha'_L(G)$. We conclude with some sufficient conditions for $\alpha'_g(G)$ to be half the order.

1.1. Definition list

For notation and graph-theory terminology not defined herein, we in general follow [11]. We denote the *degree* of a vertex v in a graph G by $d(v)$. The minimum degree among the vertices of G is denoted by $\delta(G)$. A *matching* M is a set of edges in G no two of which are adjacent; it is *perfect* if every vertex of G is incident to an edge of M. If M is a matching in G, then a vertex is M*-matched* or *covered by* M if it is incident with an edge of M; otherwise, the vertex is M*-unmatched*. A path or cycle is M*-alternating* of its edges are alternately in and not in M.

2. The min-start game

We start by showing that the Min-start matcher number of a graph is precisely the matching number of the graph. We shall need the following trivial well-known preliminary lemma.

Lemma 1. *If G is a graph, then every non-isolated vertex is incident with an edge that belongs to some maximum matching of G.*

Proof. Suppose that M is a maximum matching and vertex u is M-unmatched but has a neighbor w. Then w is M-matched; say f is the edge of M incident with w. Then $(M \setminus \{f\}) \cup \{uw\}$ is a maximum matching in G containing an edge incident with u. □

Theorem 1. *For every graph G, we have $\hat{\alpha}'_g(G) = \alpha'(G)$.*

Proof. We show that Maximizer has a strategy that guarantees that the Min-start matcher game always finishes with a maximum matching, implying that $\hat{\alpha}'_g(G) = \alpha'(G)$. Suppose that Minimizer chooses vertex u. By Lemma 1, the vertex u is incident with an edge, say uv, that belongs to some maximum matching, M say, in G. So Maximizer chooses vertex v. Then let $G' = G - \{u, v\}$. Note that $\alpha'(G') = \alpha'(G) - 1$, while by induction we have $\hat{\alpha}'_g(G') = \alpha'(G')$. The result follows. □

3. Basics of the max-start game

For the remainder of the paper we study the Max-start matcher game. Here is an example:

Lemma 2. *For the path P_n with $n \geq 1$ it holds that $\alpha'_g(P_n) = \lfloor n/2 \rfloor$. For the cycle C_n with $n \geq 3$ it holds that $\alpha'_g(C_n) = \lfloor n/2 \rfloor$.*

Proof. The stated value is the matching number of the graphs. So it suffices to show that Maximizer can force a maximum matching. For the path, Maximizer starts with an end-vertex, and then repeatedly chooses a vertex adjacent to one that has already been matched. For the cycle, the result follows by the same strategy. □

The above case is an example where the matcher number equals the matching number. For an example where these two numbers are different, let G be the graph formed by taking two disjoint copies of the complete graph K_m, for odd $m \geq 3$, and adding one edge between the cliques. Then no matter which vertex Maximizer chooses first, Minimizer can ensure that exactly one cut-vertex is chosen; the removal of the first two chosen vertices leaves two cliques of odd order. Thus $\alpha'_g(G) = m - 1$ while $\alpha'(G) = m$.

Another case where the matcher and matching numbers are different is the following:

Lemma 3. *If G is a graph with minimum degree at least two that contains a unique maximum matching, then $\alpha'_g(G) < \alpha'(G)$.*

Proof. Let M be the unique maximum matching in G. By Lemma 1, every vertex is incident with M; that is, M is a perfect matching. Let u be the first vertex chosen by Maximizer. Since u has degree at least two, Minimizer can respond by choosing a vertex v such that uv is not in M; thus the maximal matching that is constructed will not be M. It follows that $\alpha'_g(G) < \alpha'(G)$. □

A useful concept is the following. We define a vertex v of a graph G as *liberal* if it is not isolated and every edge incident with v belongs to some maximum matching in G. The usefulness of the idea can be expressed in the following observation. Recall that a family of graphs is *hereditary* if it is closed under vertex removal.

Lemma 4. *Let $\mathcal{G}$ be a hereditary graph family such that every nonempty graph in $\mathcal{G}$ has a liberal vertex. Then for all $G \in \mathcal{G}$ it holds that $\alpha'_g(G) = \alpha'(G)$.*

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