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# The minimum size of graphs satisfying cut conditions

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ABSTRACT

A graph $G$ of order $n$ satisfies the *cut condition* (CC) if there are at least $|A|$ edges between any set $A \subset V(G)$, $|A| \leq n/2$, and its complement $\overline{A} = V(G) \setminus A$. For even $n$, $G$ satisfies the *even cut condition* (ECC), if $[A, \overline{A}]$ contains at least $n/2$ edges, for every $A \subset V(G)$, $|A| = n/2$. We investigate here the minimum number of edges in a graph $G$ satisfying CC or ECC. A simple counting argument shows that for both cut conditions $|E(G)| \geq n - 1$, and the star $K_{1,n-1}$ is extremal. Faudree et al. (1999) conjectured that the extremal graphs with maximum degree $\Delta(G) < n-1$ satisfying ECC have $3n/2 - O(1)$ edges. Here we prove the tight bound $|E(G)| \geq 3n/2-3$, for every graph $G$ with $\Delta(G) < n-1$ and satisfying CC. If $G$ is 2-connected and satisfies ECC, we prove that $|E(G)| \geq 3n/2 - 2$ holds and tight, for every even $n$. We obtain the weaker bound $|E(G)| \geq 5n/4 - 2$, for every graph of order $n \equiv 0 \pmod 4$ with $\Delta(G) < n - 1$ and satisfying ECC; meanwhile we conjecture that $|E(G)| \geq 3n/2 - 4$ holds, for every even $n$.

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## 1. Introduction

Our investigation continues the study of certain graph properties in connection with a graph theory model of a practical networking problem initiated by Csaba et al. in [2]. The transmission of data between hosts or terminals is facilitated by computer networks. A network is made up of two types of components: nodes (processors) and communication lines (links) between nodes. We assume that the communication between the hosts is based on a point-to-point protocol where a message follows a specific route across the network. Suppose that a host $S$ wishes to communicate with another host $T$. In circuit switching mode, a dedicated communication path is allocated between $S$ and $T$, via a set of intermediate nodes and this path is maintained for the duration of communication between $S$ and $T$. It is assumed that no two communication paths between distinct pairs can share common communication lines, although the paths can use common intermediate nodes. We require the network to allow messages to be passed simultaneously between any fixed number of disjoint pairs of nodes of the network. To meet this requirement the network must satisfy certain connectivity (linkage) properties. For instance, if a cluster of $k$ hosts in one part of the network wishes to communicate with $k$ hosts at another part of the network, then there must be at least $k$ communication lines leaving the cluster.

A simple graph can quite naturally represent our communication network. The requirement of simultaneous communication between disjoint pairs of terminals translates into a specific linkage property called the *path-pairability* of the graph; a graph $G$ of even order $n$ is *path-pairable*, if for any set of disjoint pairs $\{s_i, t_i\}$, $1 \leq i \leq n/2$, $G$ has pairwise edge disjoint $(s_i, t_i)$-paths, $1 \leq i \leq n/2$. The connectivity requirements involving clusters of terminals correspond to *cut conditions* to be satisfied by the graph. The basic problem is to determine and build optimal networks (addressing a minimum number of nodes, or a minimum number of communication lines) which leads to extremal problems on graphs satisfying the path-pairability

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requirement or the connectivity conditions. Since the early nineties several publications studied related extremal graph theory questions and obtained various results (see [5–8,10,13]). The property of a graph being *weakly k-linked* is the closest concept in graph theory to path-pairability; for a comparison with further linkage properties see [14]. The cut conditions are strongly related to the concept of edge isoperimetric number or edge expansion of a graph, a discrete version of the Cheeger constant, extensively studied in the literature.

Simple graphs with no loops or multiple edges are considered here. In a graph $G$ a non-empty set $A \subset V(G)$ and its complement $\overline{A} = V(G) \setminus A$ define a cut $[A, \overline{A}] \subseteq E(G)$. The *size of a cut*, denoted $|[A, \overline{A}]|$, is the number of 'cross-edges' between $A$ and $\overline{A}$. If $G$ has even order $n$, a cut $[A, \overline{A}]$ with $|A| = n/2$ is called an even cut or a *bisection*. A graph $G$ satisfies the *cut condition* (CC), if $|[A, \overline{A}]| \geq \min\{|A|, |\overline{A}|\}$, for every (non-empty) $A \subset V(G)$. A graph $G$ of even order $n$ satisfies the *even cut condition* (ECC), if $|[A, \overline{A}]| \geq n/2$, for every bisection $[A, \overline{A}]$. The cut condition can be expressed in terms of the *isoperimetric number* of a graph $G$ defined in [15] as

$$i(G) = \min_{\emptyset \neq A \subset V(G)} \frac{|[A, \overline{A}]|}{\min\{|A|, |\overline{A}|\}}.$$

In fact, $G$ satisfies CC if and only if $i(G) \geq 1$.

The communication network model described above leads to the concept of path-pairable graphs, a proper subfamily of graphs satisfying CC (see [2]). A path-pairable graph is obviously connected, moreover it satisfies CC which condition is not sufficient for a graph to be path-pairable. For a non-trivial example, the hypercube $Q^d$ is not path-pairable for $d$ even (due to easy parity reasons). On the other hand, the isoperimetric number of $Q^d$, for any $d$, was determined in [9] as a result of the isoperimetric inequality; $i(Q^d) = 1$ also follows from a more general result concerning the isoperimetric number of Cartesian products in [15] (Theorem 5.1). It is also conjectured in [2] that $Q^d$ is path-pairable, for every $d$ odd.

Further investigations of path-pairability in [7] revealed the problem of determining the minimum size (number of edges) of a path-pairable graph $G$ of (even) order $n$. Since $G$ must be connected, and the star $K_{1,n-1}$ is path-pairable, the trivial sharp bound $|E(G)| \geq n - 1$ follows. For path-pairable graphs $G$ with maximum degree $\Delta(G) < n - 1$, the bound $|E(G)| \geq 3n/2 - \log_2 n - O(1)$ was proved in [7]. Here we extend the investigations initiated in [7] from path-pairable graphs to the rudimentary families of graphs satisfying CC or ECC.

In Section 2 we prove the tight bound $|E(G)| \geq 3n/2 - 3$, for every graph $G$ of even order $n$ with $\Delta(G) < n - 1$ and satisfying CC (Theorem 2). This improves the above mentioned bound in [7] for path-pairable graphs (Corollary 1), where Faudree et al. also made the following conjecture.

**Conjecture 1** (*Faudree et al. [7]*)**.** *Let $G$ be a graph of even order $n$ with $\Delta(G) < n - 1$. If $G$ satisfies the even cut condition, then $|E(G)| \geq 3n/2 - O(1)$.*

In Section 3, Conjecture 1 is verified for some subfamilies, in particular for graphs with fixed large maximum degree. For a graph $G$ satisfying ECC with $\Delta(G) = n - 2$ or $n - 3$, we obtain $|E(G)| \geq 3n/2 - 3$ (Propositions 1, 2), the same bound which holds under (the more restrictive) CC in Theorem 2. As observed in the proof of Theorem 2 we have a bound $|E(G)| \geq 3n/2 - 2$, provided $G$ is a 2-connected graph. It is worth noting that this bound is tight for 2-connected graphs (Proposition 3). There are graphs satisfying ECC with $\Delta(G) < n - 1$ and $|E(G)| = 3n/2 - 4$, for every even $n \geq 8$; further data and observations (not included here) support an update of Conjecture 1 as follows.

**Conjecture 2.** *Let $G$ be a graph of even order $n$ with $\Delta(G) < n - 1$. If $G$ satisfies the even cut condition, then $|E(G)| \geq 3n/2 - 4$.*

In Section 4 we prove the weaker bound $|E(G)| \geq 5n/4 - 2$, for every graph of order $n \equiv 0 \pmod 4$ with $\Delta(G) < n - 1$ and satisfying ECC (Theorem 5).

## 2. Edge minimum under CC

We shall refer to well-known results on partitioning and sequencing the vertex set of a 2-connected graph of order $k$. The first theorem of interest is in Lovász [12] (§6,Ex.8(a)): if $G$ is a 2-connected graph of order $k = k_1 + k_2$, where $k_1, k_2$ are positive integers, then the vertex set of $G$ has a partition $A_1 \cup A_2$ such that $|A_i| = k_i$ and $A_i$ induces a connected subgraph, for $i = 1, 2$. In addition to Lovász' theorem we use a stronger property of 2-connected graphs, the existence of an *st-numbering*, introduced in [11] for efficient planarity testing (see also [4]).

**Lemma 1.** *If $B$ is a 2-connected graph of order $k$, and $s, t \in V(B)$, then $V(B)$ has a labeling $x_1, x_2, \ldots, x_k$ such that $x_1 = s$, $x_k = t$, and for every $1 < i < k$, $x_i$ has a neighbor in both sets $\{x_1, \ldots, x_{i-1}\}$ and $\{x_{i+1}, \ldots, x_k\}$.* □

If a connected graph $G$ is not 2-connected, then it has a *cut vertex*, a vertex whose removal disconnects the graph. The maximal 2-connected subgraphs, called *blocks* of $G$, define a partition of its edges into a tree-like *block decomposition* best described as a *block-cut vertex tree* (see in [1]).

**Theorem 2.** *Let $G$ be a graph of even order $n$ with maximum degree $\Delta(G) < n - 1$. If $G$ satisfies the cut condition, then $|E(G)| \geq 3n/2 - 3$, and this bound is sharp.*

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