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Neighbor sum distinguishing total coloring and list neighbor sum distinguishing total coloring

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ABSTRACT

Let $\chi_{\Sigma}^{t}(G)$ and $\chi_{\Sigma}^{lt}(G)$ be the neighbor sum distinguishing total chromatic and total choice numbers of a graph G, respectively. In this paper, we present some new upper bounds of $\chi_{\Sigma}^{lt}(G)$ for ℓ-degenerate graphs with integer $\ell \geq 1$, and of $\chi_{\Sigma}^{t}(G)$ for 2-degenerate graphs. As applications of these results, (i) for a general graph G, $\chi_{\Sigma}^{t}(G) \leq \chi_{\Sigma}^{lt}(G) \leq \max\{\Delta(G) + \lfloor \frac{3\mathrm{col}(G)}{2} \rfloor - 1, 3\mathrm{col}(G) - 2\}$, where $\mathrm{col}(G)$ is the coloring number of G; (ii) for a 2-degenerate graph G, we determine the exact value of $\chi_{\Sigma}^{t}(G)$ if $\Delta(G) \geq 6$ and show that $\chi_{\Sigma}^{t}(G) \leq 7$ if $\Delta(G) \leq 5$.

1. Introduction

For terminology and notations not defined here we follow [3]. Let $G = (V, E)$ be a simple graph and $\phi : V \cup E \to [k]$ be a total-$[k]$-weighting where $[k] = \{1, \dots, k\}$. ϕ is a neighbor sum distinguishing total-$[k]$-weighting (abbreviated as NSD total-$[k]$-weighting) if $\phi(u) + \sum_{e \in E(u)} \phi(e) \neq \phi(v) + \sum_{e \in E(v)} \phi(e)$ for each edge $uv \in E$, where $E(x)$ is the set of edges incident with x for a vertex x. The NSD total weighting problem was introduced by Przybyło and Woźniak [18] as a variation of a similar problem for edge weighting introduced by Karoński, Łuczak and Thomason [12]. Przybyło and Woźniak [18] conjectured that every graph has an NSD total-[2]-weighting while Karoński, Łuczak and Thomason [12] conjectured that every graph without isolated edges has an NSD edge-[3]-weighting. Those two conjectures are known as the 1–2 Conjecture and the 1–2–3 Conjecture, respectively. Kalkowski, Karoński and Pfender [11] showed that every connected graph with at least three vertices has an NSD edge-[5]-weighting. Kalkowski [10] showed that every graph has an NSD total-[3]-weighting. The readers are referred to a survey paper [20] for more information.

Recently both NSD total weighting and NSD edge weighting have been extended to choosability version. In [2], Bartnicki et al. considered the choosability version of 1–2–3 Conjecture. A graph is said to be NSD edge-k-weight-choosable if for every list assignment L which assigns to each edge e a set $L(e)$ of k real numbers, G has an NSD edge weighting ϕ such that $\phi(e) \in L(e)$ for each edge e. Przybyło and Woźniak [19] and Wong and Zhu [22] independently introduced the list version of NSD total weighting. Let $L(x)$ be a list of k real numbers assigned to each element $x \in V \cup E$ of a graph G, and let L be a collection of all these lists. Such L is called a total-k-list assignment of G. We say that G has an NSD total-L-weighting if there is an NSD total weighting $\phi : V \cup E \to \cup_{L(x) \in L} L(x)$ such that $\phi(x) \in L(x)$ for each $x \in V \cup E$. A graph G is NSD total-k-weight-choosable if for each total-k-list assignment L, G has an NSD total-L-weighting. Przybyło and Woźniak [19] conjectured that the 1–2 Conjecture is also true for the list version. That is every graph is NSD total-2-weight-choosable. This conjecture was verified

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for trees, cycles, and complete graphs by Przybyło [16] and Wong and Zhu [22] independently. In [23], Wong and Zhu proved that every graph is NSD total-3-weight-choosable.

The (list) edge coloring version of the problem where the edge weighting is a (list) edge coloring has been studied for recent years (see, for example, [7–9,17,21]). In this paper, we study the (list) total coloring version of the problem. Let $k \geq 1$ be an integer and L be a total-k-list assignment of a graph G. A total coloring ϕ is called a total-L-coloring if $\phi(x) \in L(x)$ for each $x \in V \cup E$. Such ϕ is neighbor sum distinguishing (NSD) if $\phi(u) + \sum_{e\in E(u)}\phi(e) \neq \phi(v) + \sum_{e\in E(v)}\phi(e)$ for each $uv \in E$. The NSD total choice number, denoted by $\chi^{lt}_{\Sigma}(G)$, is the smallest integer k such that for each total-k-list assignment L, G has an NSD total-L-coloring. In particular, a total-L-coloring is said to be a total-$[k]$-coloring if $L(x) = [k]$ for each $x \in V \cup E$. The NSD total chromatic number of G, denoted by $\chi^{t}_{\Sigma}(G)$, is the smallest integer k such that G has an NSD total-$[k]$-coloring. The concept of NSD total-$[k]$-coloring was introduced by Pilśniak and Woźniak [15], who proposed the following conjecture.

Conjecture 1.1 ([15]). *For a graph G with maximum degree Δ, $\chi^{t}_{\Sigma}(G) \leq \Delta + 3$.*

Pilśniak and Woźniak [15] showed that Conjecture 1.1 holds for complete graphs, bipartite graphs and graphs with maximum degree 3. Li et al. [13] proved that Conjecture 1.1 holds for planar graphs with maximum degree at least 13. Recently, Li et al. [14] proved the following result.

Theorem 1.2 ([14]). *Conjecture 1.1 holds for K_4-minor-free graphs.*

The *maximum average degree* of G is $\mathrm{mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$. Dong and Wang [6] showed the following theorem.

Theorem 1.3 ([6]). *Let G be a graph with at least two vertices. If $\mathrm{mad}(G) < 3$, then $\chi^{t}_{\Sigma}(G) \leq \max\{\Delta + 2, 7\}$.*

The coloring number of a graph G, denoted by $\mathrm{col}(G)$, is the least integer k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbors. By using combinatorial nullstellensatz, Ding et al. [5] presented an upper bound of $\chi^{t}_{\Sigma}(G)$ as follows.

Theorem 1.4 ([5]). *For a graph G with at least two vertices, $\chi^{t}_{\Sigma}(G) \leq 2\Delta + \mathrm{col}(G) - 1$.*

In this paper, we improve the upper bound in Theorem 1.4 by proving its list version.

Theorem 1.5. *For a graph G with maximum degree Δ,*

$$\chi^{t}_{\Sigma}(G) \leq \chi^{lt}_{\Sigma}(G) \leq \begin{cases} \max\{\Delta + \lfloor \frac{3\mathrm{col}(G)}{2} \rfloor - 1, 3\mathrm{col}(G) - 2\}, & \text{if } col(G) \leq 3; \\ \max\{\Delta + \lfloor \frac{3\mathrm{col}(G)}{2} \rfloor - 2, 3\mathrm{col}(G) - 2\}, & \text{if } col(G) \geq 4. \end{cases}$$

For a positive integer ℓ, G is *ℓ-degenerate* if every subgraph of G contains a vertex of degree at most ℓ. Note that every graph G is $(\mathrm{col}(G) - 1)$-degenerate. For a 2-degenerate graph G, we determine the exact value of $\chi^{t}_{\Sigma}(G)$ when $\Delta \geq 6$.

Theorem 1.6. *Let G be a 2-degenerate graph with maximum degree Δ. Then*
(1) If $\Delta \leq 5$, then $\chi^{t}_{\Sigma}(G) \leq 7$.
(2) If $\Delta \geq 6$, then

$$\chi^{t}_{\Sigma}(G) = \begin{cases} \Delta + 1, & \text{if } G \text{ contains no two adjacent } \Delta\text{-vertices}; \\ \Delta + 2, & \text{otherwise}. \end{cases}$$

Note that trees, K_4-minor-free graphs, parallel series graphs, planar graphs with girth at least 6 and graphs with $\mathrm{mad}(G) < 3$ are all 2-degenerate. Therefore, Theorem 1.6 implies Theorems 1.2 and 1.3.

Theorem 1.5 is a direct corollary of Theorem 1.7 on ℓ-degenerate graphs.

Theorem 1.7. *Let $\ell \geq 1$ be an integer and G be an ℓ-degenerate graph with maximum degree Δ. Then*

$$\chi^{lt}_{\Sigma}(G) \leq \begin{cases} \max\{\Delta + \lfloor \frac{3\ell+1}{2} \rfloor, 3\ell + 1\}, & \text{if } \ell \leq 2; \\ \max\{\Delta + \lfloor \frac{3\ell-1}{2} \rfloor, 3\ell + 1\}, & \text{if } \ell \geq 3. \end{cases}$$

We believe that the list version of Conjecture 1.1 is also true.

Conjecture 1.8. *For a graph G with maximum degree Δ, $\chi^{lt}_{\Sigma}(G) \leq \Delta + 3$.*

Theorem 1.7 implies that Conjecture 1.8 is true for 2-degenerate graphs with $\Delta \geq 4$. Note that if $\Delta = 1$ then $\chi^{t}_{\Sigma}(G) = \chi^{lt}_{\Sigma}(G) = 3$, and if G contains two adjacent Δ-vertices then $\chi^{lt}_{\Sigma}(G) \geq \chi^{t}_{\Sigma}(G) \geq \Delta+2$. Since all trees are 1-degenerate, by Theorem 1.7, we have the following corollary.

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