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Note

# Tight bounds on the complexity of semi-equitable coloring of cubic and subcubic graphs☆

Hanna Furmańczyk [a,*], Marek Kubale [b]

[a] Institute of Informatics, University of Gdańsk, Wita Stwosza 57, 80-308 Gdańsk, Poland
[b] Department of Algorithms and System Modelling, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland

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A B S T R A C T

A $k$-coloring of a graph $G = (V, E)$ is called semi-equitable if there exists a partition of its vertex set into independent subsets $V_1, \ldots, V_k$ in such a way that $|V_1| \notin \{\lceil |V|/k \rceil, \lfloor |V|/k \rfloor\}$ and $||V_i| - |V_j|| \leq 1$ for each $i, j = 2, \ldots, k$. The color class $V_1$ is called non-equitable. In this note we consider the complexity of semi-equitable $k$-coloring, $k \geq 4$, of the vertices of a cubic or subcubic graph $G$. In particular, we show that, given a $n$-vertex subcubic graph $G$ and constants $\epsilon > 0$, $k \geq 4$, it is NP-complete to obtain a semi-equitable $k$-coloring of $G$ whose non-equitable color class is of size $s$ if $s \geq n/3 + \epsilon n$, and it is polynomially solvable if $s \leq n/3$.



## 1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. We refer the reader to [11] for terminology in graph theory. We say that a graph $G = (V, E)$ is *equitably k-colorable* if and only if its vertex set can be partitioned into independent sets $V_1, \ldots, V_k \subset V$ such that $|V_i| - |V_j| \in \{-1, 0, 1\}$ for all $i, j = 1, \ldots, k$. The smallest $k$ for which $G$ admits such a coloring is called the *equitable chromatic number* of $G$ and denoted by $\chi_=(G)$. A graph $G$ on $n$ vertices has a *semi-equitable coloring* if there exists a partition of its vertices into independent sets $V_1, \ldots, V_k \subset V$ such that one of these subsets, say $V_1$, is of size $s \notin \{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$, and the remaining subgraph $G - V_1$ is equitably $(k-1)$-colorable. In what follows, such a color class $V_1$ will be called *non-equitable*. These two models of graph coloring are motivated by applications in multiprocessor scheduling of unit-execution time jobs [7,6].

In the following we will say that graph $G$ has a $(V_1, \ldots, V_k)$-coloring to express explicitly the partition of $V$ into $k$ independent sets. If, however, only cardinalities of color classes are important, we will use the notation of $[|V_1|, \ldots, |V_k|]$-coloring. For a given coloring, we call the difference $\max\{|V_i| - |V_j| : i, j \in \{1, \ldots, k\}\}$ its *color width*. Thus, a coloring of a graph is equitable if and only if the color width does not exceed 1.

We mention the following two theorems on equitable graph coloring. First, Hajnal and Szemeredi [9] proved.

**Theorem 1.1** ([9]). *If $G$ is a graph with maximum degree $\Delta(G)$, $\Delta(G) \leq k$, then $G$ has an equitable $(k+1)$-coloring.*

This theorem implies that every cubic graph, i.e. a regular graph of degree 3, has an equitable $k$-coloring for every $k \geq 4$. Kierstead et al. [10] gave a simple algorithm for obtaining such a coloring in $O(n^2)$ time. Secondly, Chen et al. [1] proved.

**Theorem 1.2** ([1]). *If $G$ is a connected 3-chromatic cubic graph, then there exists an equitable 3-coloring of $G$.*

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* Corresponding author.
*E-mail addresses:* hanna@inf.ug.edu.pl (H. Furmańczyk), kubale@eti.pg.edu.pl (M. Kubale).

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Actually, they proved that $\chi(G) = \chi_=(G)$ for any connected cubic graph $G$. The proof starts from any proper 3-coloring of a connected cubic graph different from $K_4$ and $K_{3,3}$, and it relies on successive decreasing of the color width of this coloring by one or by two, step by step, until the coloring is equitable. Moreover, Chen and Yen in [2] extended this result to disconnected subcubic graphs, where by a *subcubic graph* we mean a graph $G = (V, E)$ with $\deg(v) \leq 3$ for all $v \in V$.

**Theorem 1.3** ([2]). *A subcubic graph $G$ with $\chi(G) \leq 3$ is equitably 3-colorable if and only if exactly one of the following statements holds.*

1. *No components or at least two components of $G$ are isomorphic to $K_{3,3}$.*
2. *Only one component of $G$ is isomorphic to $K_{3,3}$ and $\alpha(G - K_{3,3}) > \frac{|V(G-K_{3,3})|}{3} > 0$.*

By the above we immediately have the following corollary.

**Corollary 1.4.** *If $G$ is a subcubic graph including neither $K_{3,3}$ nor $K_4$ as a component, then it admits an equitable 3-coloring.* □

The problem of semi-equitable 3-coloring of connected cubic graphs was introduced in [4]. We have shown that every cubic graph with $t$ independent vertices has equitable 3-coloring for $t \in \{\lceil n/3 \rceil, \lfloor n/3 \rfloor\}$ and semi-equitable 3-coloring for $t \geq 2n/5$. In this note we extend those results to an arbitrary number $k \geq 4$ of colors and to, possibly disconnected, subcubic graphs. In contrast to equitable coloring not all cubic/subcubic graphs have a semi-equitable coloring (see $K_4$ for example). Therefore, in the following we assume that all graphs under consideration have such a coloring. We will denote by $N(v)$ *the (open) neighborhood* of the vertex $v \in V$, that is the set $\{u \in V : \{u, v\} \in E\}$. Let $G_1 \cup G_2$ denote the *union of graphs* $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$, i.e. the graph $G$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

**Theorem 1.5** ([2]). *If two graphs $G_1$ and $G_2$ with disjoint vertex sets are both equitably $k$-colorable, then $G_1 \cup G_2$ is also equitably $k$-colorable.*

The rest of the paper is organized as follows. In Section 2 we prove that, in contrast to equitable coloring, the problem of semi-equitable coloring becomes NP-complete for each $k \geq 3$. More precisely, we show that computing a semi-equitable $k$-coloring of a subcubic graph whose maximum color class is of size at least $(1/3 + \epsilon)n$ for any $\epsilon \in (0, 1/6)$ is NP-complete, if one exists. In Section 3 we show how to obtain in $O(n^2)$ time a semi-equitable $k$-coloring of a subcubic graph with non-equitable color class of size at most $n/3$. Because we are interested in an algorithmic approach, in Appendix we reprove Corollary 1.4 by giving an appropriate algorithm resulting from a slight modification of Chen et al.'s proof [1]. The computational complexity of the whole equalizing procedure is $O(n^2)$.

## 2. NP-completeness of the problem

In this section we present one of the main results of the paper. We are interested in the computational complexity of deciding whether a subcubic graph $G$ has a semi-equitable $k$-coloring ($k \geq 4$) with non-equitable color class of size $s$. In [5] we proved that the problem of deciding whether a cubic graph has a coloring of type $[4n/10, 3n/10, 3n/10]$ is NP-complete. In the following we strengthen and generalize this result to semi-equitable $k$-colorings ($k \geq 4$) and subcubic graphs. We consider graphs not including $K_4$ as a component. We say that $G$ is a *m-divisible* graph if $|V(G)|$ is divisible by $m$.

Let us define the following decision problem.

$(\frac{1}{3} + \epsilon)$*-Semi-Equitable Coloring of a Cubic m-Divisible graph* (SECCD).
Instance: A $m$-divisible cubic graph $G$, an integer $k \geq 4$, and $\epsilon > 0$.
Question: Does $G$ have a semi-equitable $k$-coloring whose non-equitable color class is of size at least $(\frac{1}{3} + \epsilon)|V|$?

We want to prove that SECCD is NP-complete. The following lemma states a strong relationship between our problem and the Independent Set Problem.

**Lemma 2.1.** *Let $G$ be a cubic or subcubic graph, $k, s \in \mathbb{Z}^+$, and $k \geq 4$. Then $G$ has a semi-equitable $k$-coloring with non-equitable color class of size $s$ if and only if $G$ has an independent set of size $s$.*

**Proof.** If $G$ has a semi-equitable $k$-coloring, $k \geq 4$, with non-equitable color class of size $s$ then there must exist an independent set of size $s$ in $G$.

Conversely, if there is an independent set of size $s$ in $G$, it forms a non-equitable color class, say $V_1$. If $k \geq 5$, the existence of an equitable $(k-1)$-coloring of the remaining subcubic graph $G - V_1$ follows from Theorem 1.1. Let $k = 4$. If $G - V_1$ fulfills the condition from Theorem 1.3, then we have an equitable 3-coloring of $G - V_1$. Let us assume that $G - V_1$ is not equitably 3-colorable. This means that there is only one component of $G - V_1$ isomorphic to $K_{3,3}$ and $\alpha(G - V_1 - K_{3,3}) \leq \frac{|V(G)|-s-6}{3}$ due to Theorem 1.3. Then we try to exchange one of vertices from $V_1$ to another one in the copy of $K_{3,3}$ in $G - V_1$. If we succeed, there would be no component isomorphic to $K_{3,3}$ in $G - V_1$. Otherwise we conclude that exactly $s$ vertices from $V_1$ belong to $s$ subgraphs isomorphic to $K_{3,3}$ in $G$ and the subgraph $G - V_1$ can be expressed as $sK_{2,3} \cup K_{3,3} \cup H$, where $H$ is a subgraph (possibly empty) of $G - V_1$ that is free from $K_{3,3}$. If $V(H)$ includes at least one vertex, we exchange one vertex from $V_1$ with any

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