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On the secure domination numbers of maximal outerplanar graphs

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ABSTRACT

A subset S of vertices in a graph G is a secure dominating set of G if S is a dominating set of G and, for each vertex $u \notin S$, there is a vertex $v \in S$ such that uv is an edge and $(S \setminus \{v\}) \cup \{u\}$ is also a dominating set of G. We show that if G is a maximal outerplane graph of n vertices, then G has a secure dominating set of size at most $\lceil 3n/7 \rceil$. Moreover, if a maximal outerplane graph G has no internal triangles, it has a secure dominating set of size at most $\lceil n/3 \rceil$. Finally, we show that any secure dominating set of a maximal outerplane graph without internal triangles has more than $n/4$ vertices.

1. Introduction

Let G be a simple graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The *open neighborhood* of a vertex $v \in V(G)$ is defined by $N_G(v) = \{u \mid vu \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N(u) \cup \{v\}$. We denote by $\deg_G v = |N_G(v)|$ the degree of v. The minimum degree of G is denoted by $\delta(G)$. For a subset $U \subseteq V(G)$, the subgraph induced by U is denoted by $G[U]$. For a proper subset $U \subset V(G)$, we denote by $G - U$ the graph obtained by removing vertices in U and their incident edges.

A vertex v is said to *dominate* itself and each vertex in $N_G(v)$, that is, v dominates the vertices in $N_G[v]$. A set $S \subseteq V(G)$ is a *dominating set* of G if each vertex $u \in V(G) \setminus S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Let S be a dominating set of G. A vertex $v \in S$ is said to *defend* $u \notin S$ if $uv \in E(G)$ and $S' = (S \setminus \{v\}) \cup \{u\}$ is also a dominating set. A dominating set S is a *secure dominating set* (or an *SDS*) if, for every $u \in V(G) \setminus S$, there exists a vertex $v \in S$ such that v defends u. The *secure domination number* $\gamma_s(G)$ of G is the smallest cardinality of a secure dominating set of G.

The notion of secure domination has been researched extensively. Cockayne et al. [7] investigated some fundamental properties of an SDS, and obtained exact values of $\gamma_s(G)$ for some graph classes, such as paths, cycles, complete multipartite graphs. Furthermore, various aspects of secure domination have been researched in [3,4,6,10,12,13,15–17]. The secure domination problem is NP-hard even when restricted to bipartite graphs and split graphs [15]. Burger, de Villiers, and van Vuuren proposed two exponential-time algorithms (a branch-and-bound algorithm and a branch-and-reduce algorithm) for finding a minimum SDS of an arbitrary graph [1]. Burger, de Villiers and van Vuuren [2] showed that the secure domination number of a tree is computed in linear time.

A graph G is *outerplanar* if it has embedding in the plane such that all vertices belong to the boundary of its outer face (the unbounded face). An outerplanar graph G is *maximal* if $G + uv$ is not outerplanar for any two nonadjacent vertices u and v.

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Matheson and Tarjan [11] proved a tight upper bound for the domination number on the class of *triangulated disks*: graphs that have an embedding in the plane such that all of their faces are triangles, except possibly one. They proved that $\gamma(G) \leq n/3$ for any n-vertex triangulated disk, and also showed that this bound is tight. For maximal outerplanar graphs, better upper bounds are obtained. Campos and Wakabayashi [5] showed that if G is a maximal outerplanar graph of n vertices, then $\gamma(G) \leq (n+k)/4$ where k is the number of vertices of degree 2. Tokunaga proved the same result independently in [18]. Li, Zhu, Shao and Xu improved the result by showing that $\gamma(G) \leq (n+k)/4$, where k is the number of pairs of consecutive degree 2 vertices with distance at least 3 on the outer cycle. It was investigated the *total domination* for maximal outerplanar graphs in [8,9].

In this paper, we consider the secure domination for maximal outerplanar graphs, and prove simple upper bounds for $\gamma_s(G)$, and these bounds are tight. Then, we also show a lower bound for $\gamma_s(G)$ when an outerplanar graph has an embedding such that it does not have internal triangles.

2. Secure domination

Let S be a dominating set of G, and $u \in V(G) \setminus S$. For a vertex $v \in S$, a vertex u is an *external private neighbor of* v *with respect to* S if $N(u) \cap S = \{v\}$. We denote by $\mathrm{epn}_G(v, S)$ the set of all external private neighbors of v with respect to S. Cockayne et al. [7] proved a fundamental property of a secure dominating set.

Theorem 2.1 ([7]). *Let S be a dominating set of G. A vertex $v \in S$ defends $u \in V(G) \setminus S$ if and only if $G[\mathrm{epn}_G(v, S) \cup \{u, v\}]$ is complete.*

The next proposition was given in [3].

Proposition 2.2 ([3]). *If G is a connected graph with minimum degree at least 2, there exists a minimum secure dominating set S of G such that every $v \in S$ defends some vertex.*

Upper bounds for the secure domination number of general graphs have been obtained. Burger, Henning and van Vuuren [4] provided an upper bound for a graph G with $\delta(G) \geq 2$.

Theorem 2.3 ([4]). *If G is a connected graph with $\delta(G) \geq 2$ that is not isomorphic to a cycle of length 5, then $\gamma_s(G) \leq n/2$.*

A set $S \subseteq V(G)$ is *independent* if no two vertices in S are adjacent. The maximum cardinality of an independent set of G is the *independence number* $\beta_0(G)$ of G. Merouane and Chellali [15] proved a sharp upper bound for $\gamma_s(G)$.

Theorem 2.4 ([15]). *For any graph G, $\gamma_s(G) \leq \gamma(G) + \beta_0(G) - 1$.*

For paths and cycles, the exact values of the secure domination numbers are known [7].

Theorem 2.5 ([7]). *For any path P_n of n vertices, $\gamma_s(P_n) = \lceil 3n/7 \rceil$.*

Theorem 2.6 ([7]). *For any cycle C_n of n vertices, $\gamma_s(C_3) = 1$ and $\gamma_s(C_n) = \lceil 3n/7 \rceil$ for $n \geq 4$.*

Recently, Li, Shao, and Xu [13] showed a sharp bounds for trees. A *leaf* is a vertex of degree 1, and a *stem* is a vertex adjacent to a leaf.

Theorem 2.7 ([13]). *For any tree T of $n \geq 3$ vertices,*

$$\frac{n+2}{2} \leq \gamma_s(T) \leq \frac{2n+2\ell-t}{2},$$

and the bounds are sharp, where ℓ and t are the numbers of leaves and stems of T, respectively.

3. Upper bound for maximal outerplanar graphs

A maximal outerplanar graph G can be embedded in the plane such that the boundary of the outer face is a Hamiltonian cycle and each inner face is a triangle. A maximal outerplanar graph embedded in the plane is called a *maximal outerplane graph*. For such an embedding of G, we denote by H_G the (unique) Hamiltonian cycle which is boundary of the outer face. An inner face of a maximal outerplane graph G is an *internal triangle* if it is not adjacent to outer face. A maximal outerplane graph without internal triangles is called *stripped*.

Since any maximal outerplane graph has a Hamiltonian cycle, we obtain an upper bound from Theorem 2.6.

Theorem 3.1. *For any maximal outerplane graph G with $n \geq 3$ vertices, $\gamma_s(G) \leq \lceil 3n/7 \rceil$.*

We show that there exist infinitely many maximal outerplane graphs achieve the upper bound. Let $\mathcal{G}$ denote the family of all maximal outerplane graphs G that satisfy the following conditions.

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