Discrete Applied Mathematics 236 (2018) 30–41

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The list distinguishing number of Kneser graphs

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ARTICLE INFO

Article history:
Received 11 February 2016
Received in revised form 15 September 2017
Accepted 12 October 2017
Available online 11 November 2017

Keywords:
List distinguishing number
Distinguishing number
Kneser graphs

ABSTRACT

A graph G is said to be k-distinguishable if every vertex of the graph can be colored from a set of k colors such that no non-trivial automorphism fixes every color class. The distinguishing number $D(G)$ is the least integer k for which G is k-distinguishable. If for each $v \in V(G)$ we have a list $L(v)$ of colors, and we stipulate that the color assigned to vertex v comes from its list $L(v)$ then G is said to be $\mathcal{L}$-distinguishable where $\mathcal{L} = \{L(v)\}_{v \in V(G)}$. The list distinguishing number of a graph, denoted $D_l(G)$, is the minimum integer k such that every collection of lists $\mathcal{L}$ with $|L(v)| = k$ admits an $\mathcal{L}$-distinguishing coloring. In this paper, we prove that $D_l(G) = D(G)$ when G is a Kneser graph.

1. Introduction

Let G be a graph and let $Aut(G)$ denote the full automorphism group of G. By an r-vertex coloring of G, we shall mean a map $f : V(G) \to \{1, 2, \ldots, r\}$, and the sets $f^{-1}(i)$ for $i \in \{1, 2 \ldots, r\}$ shall be referred to as the color classes of f. An automorphism $\sigma \in Aut(G)$ is said to fix a color class C of f if $\sigma(C) = C$, where $\sigma(C) = \{\sigma(v) : v \in C\}$. A vertex coloring of the graph G with the property that no non-trivial automorphism of G fixes all the color classes is called a distinguishing coloring of the graph G.

Albertson and Collins [2] defined the distinguishing number of graph G, denoted $D(G)$, as the minimum r such that G admits a distinguishing r-vertex coloring.

An interesting variant of the distinguishing number of a graph, due to Ferrara, Flesch, and Gethner [5] goes as follows. Given an assignment $\mathcal{L} = \{L(v)\}_{v \in V(G)}$ of lists of available colors to vertices of G, we say that G is $\mathcal{L}$-distinguishable if there is a distinguishing coloring f of G such that $f(v) \in L(v)$ for all v. The list distinguishing number of G, denoted $D_l(G)$, is the minimum integer k such that G is $\mathcal{L}$-distinguishable for any list assignment $\mathcal{L}$ with $|L(v)| = k$ for all v. The list distinguishing number has generated a bit of interest recently (see [5,6,8] for some relevant results) primarily due to the following question that appears in [5]:

Is$D_l(G) = D(G)$ for all graphs G?

As they state themselves, one of the authors of [5] believes this to be the case, while another author was more circumspect about the same. The authors of [5] prove the same for cycles of size at least 6, cartesian products of cycles, and for graphs whose automorphism group is a dihedral group D_{2n}. The paper [6] settles this question in the affirmative for trees, and [8] establishes it for interval graphs.

Let $r \geq 2$, and $n \geq 2r+1$. The Kneser graph $K(n, r)$ is defined as follows: The vertex set of $K(n, r)$ consists of all r-element subsets of $[n]$; vertices u, v in $K(n, r)$ are adjacent if and only if $u \cap v = \emptyset$. The distinguishing number of the Kneser graphs is well known (see [2,1]): $D(K(n, r)) = 2$ when $n \neq 5$ and $r \geq 2$; for Petersen graph $D(K(5, 2)) = 3$.

Our main result in this paper settles the aforementioned question in the affirmative for the family of Kneser graphs.

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https://doi.org/10.1016/j.dam.2017.10.006
0166-218X/

Theorem 1. *$D_l(K(n, r)) = D(K(n, r))$ for all $r \geq 2$, $n \geq 2r + 1$.*

Before we proceed to the proof of the theorem, we describe the main idea of the proof. We choose randomly (uniformly) and independently for each vertex v, a color from its list $L(v)$, and we calculate/bound the expected number of non-trivial automorphisms that fix every color class for this random set of choices. This line of argument features in some other related contexts, for e.g., [3,4,9,11] most notably under the umbrella of what is called the 'Motion Lemma', and some of its variants. For $r = 2$, the cases $8 \leq n \leq 22$ include some explicit computation using a SAGE code. These methods however do not work in the case $r = 2$ and $n = 6$ or $n = 7$, so we need different arguments to settle this case. As it turns out, the case with $r \geq 3$ is simpler than the case $r = 2$.

The rest of the paper is organized as follows. In the next couple of sections, we detail the proof for $r = 2$. The case $r \geq 3$ is considered in Section 4. We conclude with a few remarks and a conjecture in the final section. We also include an Appendix that provides the details of the SAGE code and related calculations that settle the proof for $r = 2, 8 \leq n \leq 22$.

2. List distinguishing number of $K(n, 2)$ when $n \geq 8$

As mentioned in the introduction, the distinguishing number of Kneser graphs is known [1]:

Theorem 2. *$D(K(n, 2)) = 2$ for $n \geq 6$, and $D(K(5, 2)) = 3$.*

Let S_n denote the symmetric group on n symbols. Observe that every permutation $\sigma \in S_n$ induces an automorphism of $K(n, r)$ as follows: If $v = \{i_1, i_2, \ldots, i_r\}$, then $\sigma(v) := \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)\}$. Hence S_n is contained in the full automorphism group of $K(n, r)$. If $n \geq 2r+1$, it is a well known consequence (see [7], Lemma 7.8.2, pg. 147 for instance) of the Erdős-Ko-Rado theorem that S_n is in fact the full automorphism group of $K(n, r)$.

Note that the Kneser graph $K(n, 2)$ is the complement of the line graph of K_n, so a list distinguishing coloring of the vertices of $K(n, 2)$ is easier to understand as a coloring of the edges of K_n. It is also quite straightforward to see that $D(K(n, 2)) = 2$ for $n \geq 6$. Indeed, for each $n \geq 6$, there exists a graph on n vertices with a trivial automorphism group. Fix such a graph G, color the edges of G red (say), and color the remaining edges of K_n blue (say). If $\sigma \in S_n$ is an automorphism of $K(n, 2)$ that fixes both these color classes, then in particular, σ also acts as an automorphism of G as well as its complement $\overline{G}$. But this implies that σ is the identity map. The same argument also extends to the Kneser graph $K(n, 3)$ for $r \geq 3$. However, this argument fails when the color of each vertex of $K(n, r)$ has to be an element of the list of colors assigned to v.

Suppose $n \geq 6$ and suppose $\{L(e)\}_{e \in E(K_n)}$ is a collection of lists of colors of size 2 for the edges of K_n. For each edge of K_n we choose a color uniformly and independently at random from its given list of colors. We shall refer to this as the random coloring of $K(n, r)$ in the rest of the paper. As mentioned in the introduction, we seek to compute the expected number of non-trivial automorphisms that fix all the colors class of this random coloring.

First, we set up some notations.

a. If the disjoint cycle decomposition of a permutation $\sigma \in S_n$ consists of l_i cycles of length λ_i, for $i = 1, 2, \ldots, t$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_t$, then we say σ is of type Λ where $\Lambda := (\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots, \lambda_t^{l_t})$. Note that $\sum_i l_i \lambda_i = n$.
b. $CT^{(n)}$ shall denote the set of all permutation types in S_n, i.e.,

$$CT^{(n)} := \{(\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots, \lambda_t^{l_t}) \text{ with } \sum_i l_i \lambda_i = n \text{ and } \lambda_1 < \lambda_2 < \cdots < \lambda_t\}.$$

c. $CT^{(n)}_{\geq r}$, $CT^{(n)}_r$ shall denote the sets of all permutation types with minimum cycle length at least r, and with minimum cycle length exactly r, respectively.
d. For positive integers a, b, we shall denote by (a, b) the g.c.d. of a and b.
e. $g(x) := \left\lfloor \frac{(x-1)^2}{2} \right\rfloor$ and $g(x, y) := xy - (x, y)$. Here, the functions $g(x)$ and $g(x, y)$ are defined for non-negative integers x, y.

First, observe that if a non-trivial automorphism σ fixes each of the color classes (as sets) in the random coloring of $E(K_n)$, then every edge in the orbit of an edge $e \in E(K_n)$ under the action of σ must be assigned the same color. In particular, one can compute an upper bound for the probability that σ preserves every color class as a function of the permutation type of σ.

Our current goal is the following: For a non-trivial $\sigma \in S_n$, we seek an upper bound $P(\sigma)$ on the probability that σ fixes all the color classes (as sets) in the random coloring. We then set $P(\Lambda) := \sum_{\sigma \text{ of type } \Lambda} P(\sigma)$.

Lemma 3. *Let $\sigma \in S_n$ be a non-trivial permutation of type $\Lambda = (\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots, \lambda_t^{l_t})$. Furthermore, for $i \leq j$ let $l_j^*(i) := l_i(l_i - 1)/2$ when $i = j$ and $l_j^*(i) = l_i l_j$ for all $j > i$. Then the probability that σ fixes every color class in a random coloring of $K(n, 2)$ is at most*

$$P(\sigma) := \frac{1}{2^{\mu}},$$