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Maximum Weight Independent Sets for (P_7,triangle)-free graphs in polynomial time

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ABSTRACT

The Maximum Weight Independent Set (MWIS) problem on finite undirected graphs with vertex weights asks for a set of pairwise nonadjacent vertices of maximum weight sum. MWIS is one of the most investigated and most important algorithmic graph problems; it is well known to be NP-complete, and it remains NP-complete even under various strong restrictions such as for triangle-free graphs. For a long time, its complexity was an open problem for P_k-free graphs, $k \geq 5$.

Recently, Lokshtanov et al. (2014) proved that MWIS can be solved in polynomial time for P_5-free graphs, Lokshtanov et al. (2015) proved that MWIS can be solved in quasi-polynomial time for P_6-free graphs, and Bacsó et al. (2016), and independently, Brause (2017) extended this to P_k-free graphs for every fixed k. Then very recently, Grzesik et al. (2017) showed that MWIS can be solved in polynomial time for P_6-free graphs. It still remains an open problem for P_k-free graphs, $k \geq 7$.

In this paper, we show that MWIS can be solved in polynomial time for (P_7,triangle)-free graphs. This extends the corresponding result for (P_6,triangle)-free graphs and may provide some progress in the study of MWIS for P_7-free graphs such as the recent result by Maffray and Pastor (2016) showing that MWIS can be solved in polynomial time for (P_7,bull)-free graphs.

1. Introduction

Let G be a finite, simple and undirected graph and let $V(G)$ (respectively, $E(G)$) denote the vertex set (respectively, the edge set) of G. For $U \subseteq V(G)$, let $G[U]$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood as induced subgraphs.

For $v \in V(G)$, let $N(v) := \{u \in V(G) \setminus \{v\} : uv \in E(G)\}$ be the *open neighborhood* of v in G, let $N[v] := N(v) \cup \{v\}$ be the *closed neighborhood* of v in G, and let $A(v) := V(G) \setminus N[v]$ be the *anti-neighborhood* of v in G. For $v \in V(G)$ and $U \subseteq V(G)$, with $v \notin U$, let $N_U(v) := N(v) \cap U$.

If $u \in N(v)$ ($u \notin N(v)$, respectively) we say that u *sees* v (u *misses* v, respectively). An *independent set* (or *stable set*) in a graph G is a subset of pairwise nonadjacent vertices of G. An independent set in a graph G is *maximal* if it is not properly contained in any other independent set of G.

Given a graph G and a weight function w on $V(G)$, the Maximum Weight Independent Set (MWIS) problem asks for an independent set of G with maximum weight. Let $\alpha_w(G)$ denote the maximum weight of an independent set of G. The MWIS problem is called *MIS problem* if all vertices v have the same weight $w(v) = 1$.

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The MIS problem ([GT20] in [20]) is well known to be NP-complete [24]. While it is solvable in polynomial time for bipartite graphs (see e.g. [1,17,22]), it remains NP-hard even under various strong restrictions, such as for triangle-free graphs [47].

The following specific graphs are subsequently used. P_k has vertices $v_1, v_2, \ldots, v_k$ and edges v_jv_{j+1} for $1 \leq j < k$. C_k has vertices $v_1, v_2, \ldots, v_k$ and edges v_jv_{j+1} for $1 \leq j \leq k$ (index arithmetic modulo k). K_ℓ has ℓ vertices which are pairwise adjacent. K_3 is also called *triangle*. A *claw* has vertices a, b, c, d and edges ab, ac, ad. $S_{i,j,k}$ is the graph obtained from a claw by subdividing respectively its edges into i, j, k edges (e.g., $S_{0,1,2}$ is a P_4, $S_{1,1,1}$ is a claw).

For a given graph F, a graph G is *F-free* if no induced subgraph of G is isomorphic to F. If for given graphs $F_1, \ldots, F_k$, G is F_i-free for all $1 \leq i \leq k$ then we say that G is $(F_1, \ldots, F_k)$*-free*.

Alekseev [2,5] proved that, given a graph class $\mathcal{X}$ defined by forbidding a finite family $\mathcal{F}$ of induced graphs, the MIS problem remains NP-hard for the graph class $\mathcal{X}$ if each graph in $\mathcal{F}$ is not an $S_{i,j,k}$ for some index i, j, k. Various authors [19,35,43,44,51] proved that MWIS can be solved for claw-free (i.e., $S_{1,1,1}$-free) graphs in polynomial time (improving the time bounds step by step). Lozin and Milanič [29] proved that MWIS can be solved for fork-free graphs (i.e., $S_{1,1,2}$-free graphs) in polynomial time — Alekseev [3,4] previously proved a corresponding result for the unweighted case.

For P_k-free graphs, $k \geq 5$, the complexity of MWIS was a long-standing open problem. Randerath and Schiermeyer [49] first proved that MWIS can be done in subexponential time for P_5-free graphs. Lokshtanov et al. [28] recently proved that MWIS can be solved for P_5-free graphs (i.e., $S_{0,2,2}$-free graphs) in polynomial time. Lokshtanov et al. [27] proved that MWIS can be solved in quasi-polynomial time for P_6-free graphs, and Bacsó et al. [7], and independently, Brause [14] extended this to P_k-free graphs for every fixed k. Then very recently, Grzesik et al. [23] showed that MWIS can be solved in polynomial time for P_6-free graphs.

For any fixed $k \geq 7$, it still remains an open problem for P_k-free graphs whether MWIS can be solved in polynomial time. Some characterizations of P_k-free graphs (see e.g. [6,8,9,15,18,53,54]) and some progress for MWIS are known in the literature (see e.g. [10,21,25,30,38–42]) but so far did not solve the problems.

In this paper, we show that MWIS can be solved for (P_7,triangle)-free graphs in polynomial time. This extends the corresponding result for (P_6,triangle)-free graphs and may provide some progress in the study of MWIS for P_7-free graphs.

Let us recall that the class of (P_6,triangle)-free graphs has been studied in various papers (see e.g. [11,13,26,37,50]) where various structure properties have been introduced and often applied to solve MWIS for such graphs. In particular, Brandstädt et al. [11] showed that (P_6,triangle)-free graphs have bounded clique-width, which implies that a large class of NP-hard problems (including MWIS) can be very efficiently solved for such graphs. Let us mention that on the other hand, P_7-free bipartite graphs — and thus (P_7,triangle)-free graphs — have unbounded clique-width [31].

The following result is well known:

Theorem 1 (*[1,17,22]*)**.** *Let B be a bipartite graph with n vertices.*

(i) *MWIS (with rational weights) is solvable for B in time $\mathcal{O}(n^4)$ via linear programming or network flow.*
(ii) *MIS is solvable for B in time $\mathcal{O}(n^{2.5})$.*

A graph G is *nearly bipartite* if, for each $v \in V(G)$, the subgraph $G[A(v)]$ induced by its anti-neighborhood is bipartite. Obviously we have:

$$\alpha_w(G) = \max_{v \in V(G)} \{w(v) + \alpha_w(G[A(v)])\}. \tag{1}$$

Thus, by Theorem 1, the MWIS problem (with rational weights) can be solved in time $\mathcal{O}(n^5)$ for nearly bipartite graphs.

Our approach is based on a repeated application of the *anti-neighborhood approach* with respect to (1). That allows, by detecting an opportune sequence of vertices, to split and to finally reduce the problem to certain instances of bipartite subgraphs, for which the problem can be solved in polynomial time [1,17,22]. In particular, as a corollary we obtain: For every (P_7,triangle)-free graph G there is a family $\mathcal{S}$ of subsets of $V(G)$ inducing bipartite subgraphs of G, with $\mathcal{S}$ detectable in polynomial time and containing polynomially many members, such that every maximal independent set of G is contained in some member of $\mathcal{S}$. That seems to be harmonic to the result of Prömel et al. [48] showing that with "high probability", removing a single vertex in a triangle-free graph leads to a bipartite graph.

1.1. Further notations and preliminary results

For any missing notation or reference let us refer to [12]. For $U, W \subseteq V(G)$, with $U \cap W = \emptyset$, U has a *join* (a *co-join*, respectively) to W, denoted by U ① W (U ⓪ W, respectively), if each vertex in W is adjacent (is nonadjacent, respectively) to each vertex in U.

For $v \in V(G)$ and $U \subseteq V(G)$, with $v \notin U$, v *contacts* U if v is adjacent to some vertex of U; v *dominates* U if v is adjacent to all vertices of U, that is, $\{v\}$ ① U (v ① U for short); v *misses* U if v is non-adjacent to all vertices of U, that is, $\{v\}$ ⓪ U (v ⓪ U for short).

A *component of* G is a maximal connected subgraph of G. The *distance* $d_G(u, v)$ of two vertices u, v in G is the number of edges of G in a shortest path between u and v in G.

For a subgraph H of G and $k \geq 0$, a vertex $v \notin V(H)$ is a *k-vertex for H* (or *of H*) if it has exactly k neighbors in H. *H has no k-vertex* if there is no k-vertex for H.