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On the existence of vertex-disjoint subgraphs with high degree sum[☆]

Shuya Chiba [a,*], Nicolas Lichiardopol [b,1]

[a] Applied Mathematics, Faculty of Advanced Science and Technology, Kumamoto University, 2-39-1 Kurokami, Kumamoto 860-8555, Japan
[b] Lycée A. de Craponne, Salon, France

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ABSTRACT

For a graph G, we denote by $\sigma_2(G)$ the minimum degree sum of two non-adjacent vertices if G is non-complete; otherwise, $\sigma_2(G) = +\infty$. In this paper, we prove the following two results: (i) If $s_1, s_2 \geq 2$ are integers and G is a non-complete graph with $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then G contains two vertex-disjoint subgraphs H_1 and H_2 such that each H_i is a graph of order at least $s_i + 1$ with $\sigma_2(H_i) \geq 2s_i - 1$. (ii) If $s_1, s_2 \geq 2$ are integers and G is a triangle-free graph of order at least 3 with $\sigma_2(G) \geq 2(s_1 + s_2) - 1$, then G contains two vertex-disjoint subgraphs H_1 and H_2 such that each H_i is a graph of order at least $2s_i$ with $\sigma_2(H_i) \geq 2s_i - 1$. By using this result, we also give some corollaries concerning degree conditions for the existence of k vertex-disjoint cycles.

1. Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the readers to [4]. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. We denote by $d_G(v)$ the degree of a vertex v in G. If H is a subgraph of G, then $d_H(v)$ is the number of vertices in H that are adjacent to a vertex v of G. The invariant $\sigma_2(G)$ is defined to be the minimum degree sum of two non-adjacent vertices of G, i.e., $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G), u \neq v, uv \notin E(G)\}$ if G is non-complete; otherwise, let $\sigma_2(G) = +\infty$. We denote by $g(G)$ the *girth* of G, i.e., the length of a shortest cycle of G. In this paper, "disjoint" always means "vertex-disjoint". A pair (H_1, H_2) is called a *partition of* G if H_1 and H_2 are two disjoint induced subgraphs of G such that $V(G) = V(H_1) \cup V(H_2)$.

Stiebitz [14] considered the decomposition of graphs under degree constraints and proved the following result.

Theorem A (*Stiebitz [14]*). *Let $s_1, s_2 \geq 1$ be integers, and let G be a graph. If $\delta(G) \geq s_1 + s_2 + 1$, then there exists a partition (H_1, H_2) of G such that $\delta(H_i) \geq s_i$ for $i \in \{1, 2\}$.*

Kaneko [11] showed that the same holds for triangle-free graphs with minimum degree at least $s_1 + s_2$.

Theorem B (*Kaneko [11]*). *Let $s_1, s_2 \geq 1$ be integers, and let G be a graph. If $\delta(G) \geq s_1 + s_2$ and $g(G) \geq 4$, then there exists a partition (H_1, H_2) of G such that $\delta(H_i) \geq s_i$ for $i \in \{1, 2\}$.*

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* Corresponding author.
E-mail addresses: schiba@kumamoto-u.ac.jp (S. Chiba), nicolas.lichiardopol@neuf.fr (N. Lichiardopol).
1 Deceased author.

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Diwan further improved Theorem A for graphs with girth at least 5, see [5]. Bazgan, Tuza and Vanderpooten [1] gave polynomial-time algorithms that find such partitions.

The purpose of this paper is to consider σ_2-versions of Theorems A and B. More precisely, we consider the following problems.

Problem 1. Let $s_1, s_2 \geq 2$ be integers, and let G be a non-complete graph. If $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, determine whether there exists a partition (H_1, H_2) of G such that $\sigma_2(H_i) \geq 2s_i - 1$ and $|H_i| \geq s_i + 1$ for $i \in \{1, 2\}$.

Problem 2. Let $s_1, s_2 \geq 2$ be integers, and let G be a graph of order at least 3. If $\sigma_2(G) \geq 2(s_1 + s_2) - 1$ and $g(G) \geq 4$, determine whether there exists a partition (H_1, H_2) of G such that $\sigma_2(H_i) \geq 2s_i - 1$ and $|H_i| \geq 2s_i$ for $i \in \{1, 2\}$.

In Problem 1 (resp., Problem 2), if we drop the condition "$|H_i| \geq s_i + 1$ (resp., $|H_i| \geq 2s_i$)" in the conclusion, then it is an easy problem. Because, for each edge xy in a graph G satisfying the assumption of Problem 1 (resp., the assumption of Problem 2), $H_1 = G[\{x, y\}]$ and $H_2 = G - \{x, y\}$ satisfy $\sigma_2(H_1) = \infty > 2s_1 - 1$ and $\sigma_2(H_2) \geq \sigma_2(G) - 2|\{x, y\}| \geq 2s_2 - 1$. Here, for a vertex subset X of a graph G, $G[X]$ denotes the subgraph of G induced by X, and let $G - X = G[V(G) \setminus X]$. (Similarly, for the case where $s_i = 1$ for some i, we can easily solve it.)

If G_1 is a balanced complete multipartite graph with $r + 1$ (≥ 4) partite sets of size s (≥ 2), then $\sigma_2(G_1) = 2rs = 2\big((rs-r+1)+(r-1)+1\big)-2$, and we can check that G_1 contains no partitions as in Problem 1 for $(s_1, s_2) = (rs-r+1, r-1)$. Thus, the condition "$\sigma_2(G) \geq 2(s_1+s_2+1)-1$" in Problem 1 is best possible in a sense if it is true. If G_2 is a complete bipartite graph $K_{s_1+s_2-1,s_1+s_2}$, then $\sigma_2(G_2) = 2(s_1 + s_2) - 2$ and G_2 does not contain partitions as in Problem 2. Thus, G_2 shows that the condition "$\sigma_2(G) \geq 2(s_1 + s_2) - 1$" in Problem 2 is also best possible if it is true.

Before giving the main result, we introduce the outline of the proof of Theorems A and B. The proof consists of the following two steps:

Step 1: To show the existence of two disjoint subgraphs of high minimum degree, i.e., we show the existence of two disjoint subgraphs H_1 and H_2 such that $\delta(H_i) \geq s_i$ for $i \in \{1, 2\}$.

Step 2: To show the existence of two disjoint subgraphs of high minimum degree that partition $V(G)$ by using Step 1.

In particular, in the proof of Theorems A and B, Step 2 follows easily from Step 1. In fact, if G is a graph with $\delta(G) \geq s_1+s_2-1$ and G contains a pair (H_1, H_2) of disjoint subgraphs with $\delta(H_i) \geq s_i$ for $i \in \{1, 2\}$, then we can easily transform the pair into a partition of G keeping its minimum degree condition (see [14, Proposition 4]).

Considering the situation for the proof of Theorems A and B, one may approach Problems 1 and 2 by following the same steps as above. However, for the case of σ_2-versions, neither Step 1 nor Step 2 is an easy problem because we allow vertices with low degree. In fact, in the proof of Step 2 for Theorem A ([14, Proposition 4]), the assumption that every vertex has high degree plays a crucial role. At the moment, we do not know whether we can extend disjoint subgraphs of high minimum "degree sum" to a partition or not. However, we can solve Step 1 for Problems 1 and 2. The following are our main results.

Theorem 1. *Let $s_1, s_2 \geq 2$ be integers, and let G be a non-complete graph. If $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then there exist two disjoint induced subgraphs H_1 and H_2 of G such that $\sigma_2(H_i) \geq 2s_i - 1$ and $|H_i| \geq s_i + 1$ for $i \in \{1, 2\}$.*

Theorem 2. *Let $s_1, s_2 \geq 2$ be integers, and let G be a graph of order at least 3. If $\sigma_2(G) \geq 2(s_1 + s_2) - 1$ and $g(G) \geq 4$, then there exist two disjoint induced subgraphs H_1 and H_2 of G such that $\sigma_2(H_i) \geq 2s_i - 1$ and $|H_i| \geq 2s_i$ for $i \in \{1, 2\}$.*

The graphs G_1 and G_2 defined above also show that the constraints on σ_2 in Theorems 1 and 2 cannot be weakened.

In order to show Theorems 1 and 2, we actually prove slightly stronger results as follows. Here, for a graph G and an integer s, we define $V_{\leq s}(G) = \{v \in V(G) : d_G(v) \leq s\}$.

Theorem 3. *Let $s_1, s_2 \geq 2$ be integers, and let G be a non-complete graph. If $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then there exist two disjoint induced subgraphs H_1 and H_2 of G such that for each i with $i \in \{1, 2\}$, the following hold:*

(i) *$d_{H_i}(u) \geq s_i$ for $u \in V(H_i) \setminus V_{\leq s_1+s_2}(G)$.*
(ii) *$d_{H_i}(u) + d_{H_i}(v) \geq 2s_i - 1$ for $u \in V(H_i) \setminus V_{\leq s_1+s_2}(G)$ and $v \in V(H_i) \cap V_{\leq s_1+s_2}(G)$ with $uv \notin E(H_i)$.*
(iii) *$|H_i| \geq s_i + 1$.*

Theorem 4. *Let $s_1, s_2 \geq 2$ be integers, and let G be a graph of order at least 3. If $\sigma_2(G) \geq 2(s_1 + s_2) - 1$ and $g(G) \geq 4$, then there exist two disjoint induced subgraphs H_1 and H_2 of G such that for each i with $i \in \{1, 2\}$, the following hold:*

(i) *$d_{H_i}(u) \geq s_i$ for $u \in V(H_i) \setminus V_{\leq s_1+s_2-1}(G)$.*
(ii) *$d_{H_i}(u) + d_{H_i}(v) \geq 2s_i - 1$ for $u \in V(H_i) \setminus V_{\leq s_1+s_2-1}(G)$ and $v \in V(H_i) \cap V_{\leq s_1+s_2-1}(G)$ with $uv \notin E(H_i)$.*
(iii) *$|H_i| \geq 2s_i$.*

Note that if G is a graph with $\sigma_2(G) \geq 2(s_1 + s_2 + 1) - 1$, then $G[V_{\leq s_1+s_2}(G)]$ is a complete graph (see also Lemma 1(i) in Section 2.1). Therefore, for any two distinct non-adjacent vertices in such a graph G, at least one of the two vertices belongs to $V(G) \setminus V_{\leq s_1+s_2}(G)$, i.e, (i) and (ii) of Theorem 3 imply that $\sigma_2(H_i) \geq 2s_i - 1$. Thus Theorem 1 immediately follows from