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Almost disjoint spanning trees: Relaxing the conditions for completely independent spanning trees

Benoît Darties [a], Nicolas Gastineau [b,*], Olivier Togni [a]

[a] LE2I FRE2005, Arts et Métiers, Université Bourgogne Franche-Comté, F-21000 Dijon, France
[b] PSL, Université Paris-Dauphine, LAMSADE UMR CNRS, 7243, France

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The search of spanning trees with interesting disjunction properties has led to the introduction of edge-disjoint spanning trees, independent spanning trees and more recently completely independent spanning trees. We group together these notions by defining (i, j)-disjoint spanning trees, where i (j, respectively) is the number of vertices (edges, respectively) that are shared by more than one tree. We illustrate how (i, j)-disjoint spanning trees provide some nuances between the existence of disjoint connected dominating sets and completely independent spanning trees. We prove that determining if there exist two (i, j)-disjoint spanning trees in a graph G is NP-complete, for every two positive integers i and j. Moreover we prove that for square of graphs, k-connected interval graphs, complete graphs and several grids, there exist (i, j)-disjoint spanning trees for interesting values of i and j.

1. Introduction

The graphs considered are assumed to be connected, since spanning trees are only interesting for connected graphs. Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be *spanning trees* in a graph G. The spanning trees $T_1, \ldots, T_k$ are *edge-disjoint* if $\cup_{1\leq \ell<\ell'\leq k} E(T_\ell) \cap E(T_{\ell'}) = \emptyset$. A vertex is said to be an *inner vertex* in a tree T if it has degree at least 2 in T and a *leaf* if it has degree 1. We denote by $I(T)$ the set of inner vertices of tree T. The spanning trees $T_1, \ldots, T_k$ are *internally vertex-disjoint* if $I(T_1), \ldots, I(T_k)$ are pairwise disjoint. Finally, the spanning trees $T_1, \ldots, T_k$ are *completely independent spanning trees* if they are both pairwise edge-disjoint and internally vertex-disjoint.

In this paper, we introduce (i, j)-disjoint spanning trees:

Definition 1.1. Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees in a graph G. We let $I(T_1, \ldots, T_k) = \{u \in V(G) | \exists \ell, \ell'\ u \in I(T_\ell) \cap I(T_{\ell'}),\ 1 \leq \ell < \ell' \leq k\}$ be the set of vertices which are inner vertices in at least two spanning trees among $T_1, \ldots, T_k$, and we let $E(T_1, \ldots, T_k) = \{e \in E(G) | \exists \ell, \ell',\ 1 \leq \ell < \ell' \leq k,\ e \in E(T_\ell) \cap E(T_{\ell'})\}$ be the set of edges which belong to at least two spanning trees among $T_1, \ldots, T_k$. The spanning trees $T_1, \ldots, T_k$ are (i, j)-*disjoint* for two positive integers i and j, if the two following conditions are satisfied:

(i) $|I(T_1, \ldots, T_k)| \leq i$;
(ii) $|E(T_1, \ldots, T_k)| \leq j$.

* Corresponding author.
E-mail addresses: Benoit.Darties@u-bourgogne.fr (B. Darties), Nicolas.Gastineau@dauphine.fr (N. Gastineau), Olivier.Togni@u-bourgogne.fr (O. Togni).

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By $*$ we denote a large enough integer, i.e. an integer larger than $\max(|E(G)|, |V(G)|)$, for a graph G. Remark that $(0, 0)$-disjoint spanning trees are completely independent spanning trees and that $(*, 0)$-disjoint spanning trees are edge-disjoint spanning trees. Notice also that there are infinitely many (i, j)-disjoint trees in G, for $i \geq \gamma_c(G)$ and $j \geq |V(G)| - 1$, $\gamma_c(G)$ being the minimum size of a connected dominating set in G (one can repeat infinitely the same tree with $\gamma_c(G)$ inner vertices).

1.1. Related work

Completely independent spanning trees were introduced by Hasunuma [12] and then have been studied on different classes of graphs, such as underlying graphs of line graphs [12], maximal planar graphs [13], Cartesian product of two cycles [15], complete graphs, complete bipartite and tripartite graphs [26], variant of hypercubes [6,25] and chordal rings [27]. Moreover, determining if there exist two completely independent spanning trees in a graph G is an NP-hard problem [13]. Recently, sufficient conditions inspired by the sufficient conditions for hamiltonicity have been determined in order to guarantee the existence of two completely independent spanning trees: Dirac's condition [1] and Ore's condition [7]. Moreover, Dirac's condition has been generalized to more than two trees [4,14,18] and has been independently improved [14,18] for two trees. Also, a recent paper has studied the problem on the class of k-trees, for which the authors have proven that there exist at least $\lceil k/2 \rceil$ completely independent spanning trees [23].

For a given tree T and a given pair of vertices (u, v) of T, let $P_T(u, v)$ be the set of vertices in the unique path between u and v in T. Remark that $T_1, \ldots, T_k$ are internally vertex-disjoint in a graph G if and only if for any pair of vertices (u, v) of $V(G)$, $\cup_{1\leq \ell<\ell'\leq k} P_{T_\ell}(u, v) \cap P_{T_{\ell'}}(u, v) = \{u, v\}$. Other works on disjoint spanning trees include independent spanning trees, i.e. focus on finding spanning trees $T_1, \ldots, T_k$ rooted at the same vertex r. In independent spanning trees, for any vertex v the paths between r and v in $T_1, \ldots, T_k$ are pairwise internally vertex-disjoint, i.e. for each integers i and j, $1 \leq i < j \leq k$, $P_{T_i}(r, v) \cap P_{T_j}(r, v) = \{r, v\}$. In contrast with the notion of completely independent spanning trees, in independent spanning trees only the paths to r are considered. Thus, $T_1, \ldots, T_k$ may share common vertices or edges, which is not admissible with completely independent spanning trees. Independent spanning trees have been studied for several classes of graphs which include product graphs [24], de Bruijn and Kautz digraphs [9,16], chordal rings [20], hypercubes [31,30], Möbius cubes [32] and bijective connection networks [5]. Related works also include edge-disjoint spanning trees, i.e. spanning trees which are pairwise edge-disjoint only. Edge-disjoint spanning trees have been studied on many classes of graphs, including hypercubes [2], Cartesian product of cycles [3] and Cartesian product of two graphs [19].

Some subsets of vertices $D_1, \ldots, D_k$ of a graph G are k *disjoint connected dominating sets* if $D_1, \ldots, D_k$ are pairwise disjoint and each subset is a connected dominating set in G. There are some works about disjoint connected dominating sets that can be transcribed in terms of internally vertex-disjoint spanning trees (the disjoint connected dominating sets can be used to provide the inner vertices of internally vertex-disjoint spanning trees). The maximum number of disjoint connected dominating sets in a graph G is the *connected domatic number*. This parameter is denoted by $d_c(G)$ and has been introduced by Hedetniemi and Laskar [17] in 1984. An interesting result about connected domatic number concerns planar graphs, for which Hartnell and Rall have proven that, except K_4 (which has connected domatic number 4), their connected domatic number is bounded by 3 [11]. The problem of constructing a connected dominating set is often motivated by wireless ad-hoc networks [10,29] for which connected dominating sets are used to create a virtual backbone in the network.

1.2. Motivation and basic facts about disjoint dominating sets

Remark that $(0, *)$-disjoint spanning trees are internally vertex-disjoint, and consequently, are related to connected dominating sets. Hence, we call $(0, *)$-disjoint spanning trees, *trees induced by disjoint connected dominating sets* and we give the properties about $(0, *)$-disjoint spanning trees using, when possible, the concept of disjoint connected dominating sets. Fig. 1 illustrates how disjoint connected dominating sets are used to construct $(0, *)$-disjoint spanning trees. As we observe in the next proposition, trees induced by disjoint connected dominating sets satisfy interesting properties. First, an edge can only belong to at most two trees (Proposition 1.1(i)). Second, the paths between two non-adjacent vertices in trees induced by disjoint connected dominating sets are edge-disjoint (Proposition 1.1(ii)). Moreover, the fact that the paths between two adjacent vertices share a common edge implies that these vertices are inner vertices in different trees (Proposition 1.1(iii)). These properties illustrate the utility of disjoint connected dominating sets to broadcast a message following multiples routes in a network. For a spanning tree, an *inner edge* is an edge between two inner vertices and a *leaf edge* is an edge which is not an inner edge.

Proposition 1.1. *Let i and j be two integers, $1 \leq i < j \leq k$. Let G be a graph of order at least 3, let $T_1, \ldots, T_k$ be spanning trees induced by k disjoint connected dominating sets and let $u, v \in V(G)$. By $E_T(u, v)$ we denote the set of edges in the unique path between u and v in a tree T.*

(i) *every edge belongs to at most two trees among $T_1, \ldots, T_k$;*
(ii) *if u and v are not adjacent, then $E_{T_i}(u, v) \cap E_{T_j}(u, v) = \emptyset$;*
(iii) *if $E_{T_i}(u, v) \cap E_{T_j}(u, v) \neq \emptyset$, then $\{u, v\} \not\subseteq I(T_i)$ and $\{u, v\} \not\subseteq I(T_j)$.*