ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Edge-intersection graphs of boundary-generated paths in a grid

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ARTICLE INFO

Article history:
Received 30 November 2016
Received in revised form 14 October 2017
Accepted 16 October 2017
Available online xxxx

Keywords:
B1-EPG graphs
Boundary-generated paths
Equivalence covering number
Linear k-arboricity

ABSTRACT

Edge-intersection graphs of paths on a grid (or EPG graphs) are graphs whose vertices can be represented as simple paths on a rectangular grid such that two vertices are adjacent in the graph if and only if the corresponding paths share at least one edge of the grid. For two boundary points p and q on two adjacent boundaries of a rectangular grid $\mathcal{G}$, we call the unique single-bend path connecting p and q in $\mathcal{G}$ using no other boundary point of $\mathcal{G}$ as the path *generated* by (p, q). A path in $\mathcal{G}$ is called *boundary-generated*, if it is generated by some pair of points on two adjacent boundaries of $\mathcal{G}$. In this article, we study the edge-intersection graphs of boundary-generated paths on a grid or ∂EPG graphs. The motivation for studying these graphs comes from problems in the context of circuit layout.

We show that ∂EPG graphs can be covered by two collections of vertex-disjoint co-bipartite chain graphs. This leads us to a linear-time testable characterization of ∂EPG trees and also an almost tight upper bound on the equivalence covering number of general ∂EPG graphs. We also study the cases of two-sided ∂EPG and three-sided ∂EPG graphs, which are respectively, the subclasses of ∂EPG graphs obtained when all the boundary-vertex pairs which generate the paths are restricted to lie on at most two or three boundaries of the grid. For the former case, we give a complete characterization.

1. Introduction

Edge intersection graphs of paths on a grid (or for short EPG graphs) were first introduced by Golumbic, Lipshteyn and Stern in [13]. This is the class of graphs whose vertices can be represented as simple paths on a rectangular grid so that two vertices are adjacent if and only if the corresponding paths share at least one edge of the grid.

EPG graphs have a practical use, e.g., in the context of circuit layout setting, which may be modeled as paths (wires) on a grid. In the knock-knee layout model, two wires may either cross or bend (turn) at a common grid point, but are not allowed to share a grid-edge; that is, overlap of wires is not allowed.

1.1. B_k-EPG graphs

In [13], the authors show that every graph is an EPG graph. That is, for every graph $G = (V, E)$ there exists an EPG representation $\langle \mathcal{G}, \mathcal{P} \rangle$ where $\mathcal{P} = \{P_v : v \in V\}$ is a collection of paths on a grid $\mathcal{G}$, corresponding to the vertices of V and

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https://doi.org/10.1016/j.dam.2017.10.014
0166-218X/

Please cite this article in press as: M.C. Golumbic, et al., Edge-intersection graphs of boundary-generated paths in a grid, Discrete Applied Mathematics (2017), https://doi.org/10.1016/j.dam.2017.10.014.

satisfying: paths $P_v, P_u \in \mathcal{P}$ share a grid-edge of $\mathcal{G}$ if and only if $(v, u) \in E$. Moreover, they show that if G has n vertices and m edges, then there exists an EPG representation $\langle \mathcal{G}, \mathcal{P} \rangle$ of G in which $\mathcal{G}$ is a grid of size $n \times (n+m)$ and the paths in $\mathcal{P}$ are monotonic. As such, much of the current research today focuses on subclasses of EPG graphs, and, in particular, limiting the type of paths allowed.

A turn of a path at a grid point is called a *bend* and a graph is called a *k-bend EPG graph* (denoted B_k-EPG) if it has an EPG representation in which each path has at most k bends. It is both interesting mathematically, and justified by the circuit layout application described above, to consider subclasses of graphs, e.g., by bounding the number of bends allowed in each path.

In [4], the authors show that for any k, only a small fraction of all labeled graphs on n vertices are B_k-EPG, and that for any fixed degree $d \geq 4$, a grid size of $\Theta(n^2)$ is needed to give an EPG representation of every graph with n vertices and maximum degree d, for sufficiently large n. For example, a representation of the complete bipartite graph $K_{n/2,n/2}$ needs at least $n^2/4$ grid-edges, and [13] showed that $3n^2$ grid-edges is sufficient to represent any graph.

The class of B_0-EPG graphs is easily seen to be equivalent to the well known family of interval graphs (see [11]). B_1-EPG graphs are the single bend EPG graphs, studied further in [3,6,8,13,14]. Improving a result of [5], it was shown in [17] that every planar graph is a B_4-EPG graph. It is still open whether $k = 4$ is best possible. So far it is only known that there are planar graphs that are B_3-EPG graphs and not B_2-EPG graphs. Some subclasses of planar graphs have shown to be B_2-EPG graphs, e.g., Halin graphs [10] and outerplanar graphs [17] (thus proving a conjecture of [5]). Also, [1] have shown that circular-arc graphs are B_3-EPG graphs, and that this is best possible.

For the case of B_1-EPG graphs, Golumbic, Lipshteyn and Stern [13] showed that every tree is a B_1-EPG graph, and in [14] they showed that single bend paths on a grid have strong Helly number 4. Asinowski and Ries [3] proved that every B_1-EPG graph on n vertices contains either a clique or a stable set of size at least $n^{1/3}$. In [3], the authors also give a characterization of the B_1-EPG graphs among some subclasses of chordal graphs, namely, chordal bull-free graphs, chordal claw-free graphs, chordal diamond-free graphs, and special cases of split graphs. In [8], a characterization of the sub-family of cographs that are B_1-EPG graphs is given by a complete family of minimal forbidden induced subgraphs.

No characterization is known for B_k-EPG graphs (for any $k \geq 1$) and the recognition problems are NP-complete for $k = 1$ [16] and $k = 2$ [21]. For $k = 1$, the recognition problem remains NP-complete even if just one of the four single bend shapes is allowed, the so called L-*shaped* B_1-EPG graphs [6].

1.2. Boundary generated EPG graphs

In this paper, we consider a further restriction on B_1-EPG graphs, namely that, the endpoints of every path lie on the boundary of the host rectangular grid; see Fig. 1 for an illustration. This restriction is motivated by applications in circuit design, where it is easier to take out connections from the edge of the chip or board. This notion was first proposed for investigation in [12]. Formally,

Definition 1.1. For two boundary points p and q on two adjacent boundaries of a rectangular grid $\mathcal{G}$, we call the unique single-bend path connecting p and q in $\mathcal{G}$ using no other boundary point of $\mathcal{G}$ the path *generated* by (p, q). A path in $\mathcal{G}$ is called *boundary-generated*, if it is generated by two points on adjacent boundaries of $\mathcal{G}$. A graph G is called an *edge-intersection graph of boundary-generated paths in a grid*, ∂EPG graphs for short, if there exists a rectangular grid $\mathcal{G}$ and a representation ψ which assigns to every vertex in G, a boundary-generated path in $\mathcal{G}$ such that two vertices $u, v \in V(G)$ are adjacent in G, if and only if the corresponding paths $\psi(u)$ and $\psi(v)$ share a common grid-edge of $\mathcal{G}$. In this case, we call $\langle \mathcal{G}, \mathcal{P} \rangle$ a ∂EPG representation of G, where $\mathcal{P}$ is the multiset $\{\psi(v) : v \in V(G)\}$.

2. Preliminaries

All graphs considered are finite and undirected. The complement of a graph G is denoted by $\overline{G}$. Two adjacent (non-adjacent) vertices with the same neighborhood are called *true twins* (*false twins*). The *reduced graph* of a graph G is the graph obtained from G by deleting all but one vertex from each set of false twins. The *line graph* $L(G)$ of a graph G is the intersection graph of the edge-set of G.

An *equivalence graph* is a vertex disjoint union of cliques, or equivalently, the graph where the adjacency relation is an equivalence relation. The *equivalence covering number* $\mathrm{eq}(G)$ of a graph G is the minimum number of equivalence graphs whose union is G [2]. For triangle-free graphs, equivalence covering number is the same as edge-chromatic number.

The product dimension or Prague dimension of a graph is a parameter which is closely related to the equivalence covering number. A *product k-encoding* of a graph G is obtained by associating to each vertex v a unique vector $f(v) = (v_1, \ldots, v_k)$ over the natural numbers so that for $xy \in E(G)$ the vectors $f(x)$ and $f(y)$ differ in all coordinates and for $xy \notin E(G)$ the vectors $f(x)$ and $f(y)$ agree in at least one coordinate. The *product dimension* or *Prague dimension* of a graph G, $\mathrm{pdim}(G)$, is the smallest number k such that G has a product k-encoding. It is an easy observation (cf. [20]) that

$$\mathrm{eq}(G) \leq \mathrm{pdim}(\overline{G}) \leq \mathrm{eq}(G) + 1.$$

The difference of 1 occurs because a product k-encoding needs to associate a *unique* vector to each vertex. For instance, the product dimension of the empty graph G on two vertices is 2 whereas $\overline{G}$ can be covered by one clique. But if G has no true twins (i.e., $\overline{G}$ has no false twins), then $\mathrm{eq}(G) = \mathrm{pdim}(\overline{G})$.

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