Discrete Applied Mathematics 236 (2018) 223–234

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Partially directed snake polyominoes

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ARTICLE INFO

Article history:
Received 18 December 2016
Received in revised form 4 September 2017
Accepted 6 September 2017
Available online 24 November 2017

Keywords:
Partially directed snake polyomino
Enumeration
Generating function
Self-avoiding walk
Functional equation
Bargraph
Recurrence

ABSTRACT

We introduce and investigate the family of partially directed snake polyominoes. We first present a classification of snake polyominoes and then we turn our attention on the class of partially directed snakes. We establish recurrences, functional equations and generating functions with respect to length and height for two dimensional, three dimensional and d dimensional partially directed snake polyominoes. We then inscribe partially directed snake polyominoes in a $b \times h$ rectangle. In order to include the third variable b in our study, we trade the parameter b for two variables w, v to obtain recurrences and functional equations with four variables. We investigate subfamilies of partially directed snake polyominoes. One of these families extends to $\mathbb{Z}^d$. In particular, we show a bijective relation between the subfamily of bubble polyominoes and bargraph polyominoes.

1. Introduction

A two dimensional ($2D$) polyomino is an edge-connected set of unit cells in the square lattice up to translation. Since a polyomino can be seen as a simple graph with its square cells as vertices and its edge-contacts between adjacent squares as edges, we use the vocabulary of graph theory when we find it convenient. Thus the degree of a cell in a polyomino is the number of its adjacent cells.

A *snake polyomino* of length n, shortly referred to as a *snake*, is a polyomino that contains n cells, no cycle, such that each one of its cells is of degree at most two. A snake of length $n \geq 2$ has two cells of degree one and $n-2$ cells of degree two. Snake polyominoes are also called strip polyominoes [11]. The two cells of degree one, called the tail and the head, can be used to provide snakes with an orientation. An *oriented snake* of length $n \geq 2$ is a snake with designated tail and head cells and the orientation goes from tail to head. When the tail and head cells are not specified in a snake, we say that the snake is *non-oriented*. A snake can always be given an orientation and we will call upon this property when needed. Unless specified, snakes are considered non-oriented.

The closest combinatorial objects to oriented snake polyominoes are self-avoiding walks (see [3]) but there is no bijective relation between the two sets and we found no systematic investigation of snake polyominoes in the current literature.

It is in fact quite straightforward to see that there is an injective map ϕ from the set $\mathcal{OS}(n)$ of oriented snakes of length n to the set $\mathcal{SAW}(n)$ of self-avoiding walks with n vertices in the square lattice by simply transforming cells of snakes into vertices of walks and drawing an edge between two vertices when the two corresponding cells are edge connected (see Fig. 1(a)). This correspondence is not invertible because the minimum number of cells needed to form a U shape in a snake polyomino is superior by one to the minimum number of vertices in a U shape of a SAW. For instance Fig. 1(b) shows a SAW w that has no inverse image $\phi^{-1}(w)$. By a U shape, we mean either a snake or a self avoiding walk that goes in a given direction,

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https://doi.org/10.1016/j.dam.2017.09.003
0166-218X/

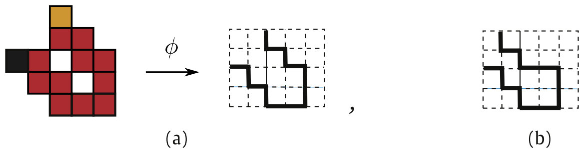

Fig. 1. An injective map from oriented snakes to self-avoiding walks of same length.

say South, then moves in a perpendicular direction, say East, then returns North, the direction opposite to the initial one, thus forming a U shape. The initial direction in a U shape can be any of the four directions. Thus, for a given length n, there are less oriented snake polyominoes than self-avoiding walks.

Despite the absence of a bijection, the complexity of the description of walks is similar to that of snakes. Walks in the plane $\mathbb{Z}^2$ have been extensively studied in the past few years (see [4] and references therein) and the methods developed in this topic can be used in the investigation of snake polyominoes.

A polyomino P is said to be inscribed in a rectangle R of size $b \times h$ with horizontal and vertical sides when P is included in R and the cells of P touch each of the four sides of R. The parameter h is then called the height of the polyomino and b, the length of the base, is called the width. Enumeration of inscribed polyominoes with area that is minimal, minimal plus one and minimal plus two were investigated in [7] and [8]. Inscribed 3D polyominoes of minimal volume were investigated in [6]. This paper is devoted to the enumeration of inscribed and non inscribed partially directed snake polyominoes, denoted *PDS*, as a part of a combinatorial study of various classes of snake polyominoes.

In Section 3 we construct the two variables generating functions of partially directed snakes with respect to length n and height h. We start with the two dimensional case, then we show that it is possible to extend the construction to the three dimensional and more generally to the d dimensional cases. In Section 4 we investigate the width b of snake polyominoes and two parameters w, v which are the horizontal distances from the head of the snake to each side of the circumscribed rectangle. We then construct a recurrence and a functional equation for the 4 variables generating function $PDS_E(x_1, x_2, y, z)$ of a subfamily PDS_E of partially directed snakes from which we can obtain a recurrence for the whole set $PDS(b, h, z)$.

In Section 5 we define the class of bubble snakes and we establish their length generating function by way of a combinatorial factorization. We show that bubble snakes are in bijection with bargraph polyominoes (see [5]). Finally we conclude with a discussion on possible extensions of this work.

2. Definitions

We first propose a classification of snake polyominoes similar to the classification of self avoiding walks (see [3]) from which we borrow part of our terminology. An oriented snake polyomino is *prudent* when, starting from the tail cell and moving towards the head cell, each direction defined by the addition of a new cell, is never done towards a cell already visited. The new cell never "sees" the previous cells of the snake. Fig. 2(a) and (b) shows, respectively, a non prudent and a prudent snake where tail is black and head is orange. Snake polyominoes possess one geometrical property that cannot exist in self-avoiding walks: two cells of a snake distant by at least two cells may touch with their corners only. When there is no such pair of cells in a snake, the snake is said to be *kiss-free*. In Fig. 2(c) and (d), the snakes are kiss-free and those in Fig. 2(a) and (b) are not. In Fig. 2(d), the head and tail are not identified with different colors to indicate that the snake is non-oriented.

A snake polyomino is a *partially directed* snake (*PDS*) towards the North when, starting from the tail cell, each new cell is added in one of the three directions North (N), East (E) or West (W) with the necessary constraint that East cannot follow immediately West and vice versa. When a W step follows an E step in a *PDS*, then at least two N steps must exist between them. The same is true when East follows West. Some snakes are prudent in one orientation and not prudent in the other but *PDS* are prudent in both orientations. Partially directed snakes have a natural orientation inherited from their construction that goes from bottom to top with the unique exception of the horizontal snake of length $n \geq 2$ which has no N step. Because of that exception, we define the set $\mathcal{OPDS}(n)$ of oriented partially directed snakes of length n as follows:

$$\mathcal{OPDS}(n) = \begin{cases} \mathcal{PDS}(n) & \text{if } n = 0, 1, \\ \mathcal{PDS}(n) \cup \{s_n\} & \text{if } n \geq 2, \end{cases} \tag{1}$$

where s_n is the horizontal snake of length n and $\mathcal{PDS}(n)$ is the set of partially directed snakes of length n so that each set $\mathcal{OPDS}(n)$, $n \geq 2$, contains two horizontal oriented snakes. We will see in Section 3 that the set $\mathcal{OPDS}$ is easier to handle than the set $\mathcal{PDS}$.

Throughout this paper we adopt the convention of using capital letters F for generating functions $F(x, y, z)$, lower case letters f for their coefficients $f(b, h, n)$

$$F(x, y, z) = \sum_{b,k,n} f(b, h, n) x^b y^h z^n,$$

and calligraphic capital letters $\mathcal{F}$ for the sets $\mathcal{F}(b, h, n)$ of cardinality $f(b, h, n)$. The set of *PDS* in $\mathbb{Z}^d$ is denoted $\mathcal{PDS}_d$ but in $\mathbb{Z}^2$, when there is no possible confusion, we simply write $\mathcal{PDS}$ and we proceed similarly for other sets of snakes. The first values of $sp(n)$, the total number of non-oriented snakes polyominoes of length n, were obtained through a computer program [9] and are presented in Table 1 together with the values $pds(n)$ (see [11]).