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Perfect Roman domination in trees

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ABSTRACT

A perfect Roman dominating function on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The weight of a perfect Roman dominating function f is the sum of the weights of the vertices. The perfect Roman domination number of G, denoted $\gamma_R^p(G)$, is the minimum weight of a perfect Roman dominating function in G. We show that if G is a tree on $n \geq 3$ vertices, then $\gamma_R^p(G) \leq \frac{4}{5}n$, and we characterize the trees achieving equality in this bound.

1. Introduction

Let $G = (V, E)$ be an undirected graph. Denote the open and closed neighborhoods of a vertex $x \in V$ by $N(x)$ and $N[x]$, respectively. That is, $N(x) = \{v \mid xv \in E\}$ and $N[x] = N(x) \cup \{x\}$. A *dominating set* of graph G is a set $D \subseteq V$ such that for each $u \in V \setminus D$, there exists an $x \in D$ adjacent to u. The minimum cardinality amongst all dominating sets of G is the *domination number*, denoted as $\gamma(G)$. A thorough treatise on dominating sets can be found in [4].

A *perfect dominating set* is a set $S \subseteq V$ such that for all $v \in V$, $|N[v] \cap S| = 1$. Perfect dominating sets and several variations on perfect domination have received much attention in the literature; for example, see some discussion in [4] or the survey in [6].

A *Roman dominating function* of a graph G, abbreviated RD-*function*, is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of a vertex v is its value, $f(v)$, assigned to it under f. The *weight*, $\mathrm{w}(f)$, of f is the sum, $\sum_{u \in V(G)} f(u)$, of the weights of the vertices. The *Roman domination number*, denoted $\gamma_R(G)$, is the minimum weight of an RD-function in G; that is,

$$\gamma_R(G) = \min\{\mathrm{w}(f) \mid f \text{ is an RD-function in } G\}.$$

Roman domination was first studied in depth in a graph theory setting in [3], after its initial introduction in the series of papers [8–11]. Roman domination was considered in trees in [5]. In this paper we introduce a perfect version of Roman domination.

A *perfect Roman dominating function* of a graph G, abbreviated PRD-*function*, is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The *perfect Roman domination number*, denoted $\gamma_R^p(G)$, is the minimum weight of a PRD-function in G; that is,

$$\gamma_R^p(G) = \min\{\mathrm{w}(f) \mid f \text{ is a PRD-function in } G\}.$$

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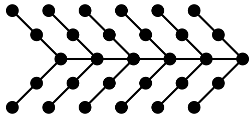

Fig. 1. A tree in the family $\mathcal{T}$.

A PRD-function with minimum weight $\gamma_R^p(G)$ in G is called a $\gamma_R^p(G)$-*function*.

Note that every graph with n vertices satisfies $\gamma_R^p(G) \le n$: this can be attained by letting $f(v) = 1$ for each vertex in the graph. As an example, let P be the Petersen graph. Then $\gamma(P) = 3$, $\gamma_R(P) = 6$ and $\gamma_R^p(P) = 7$ (the latter can be achieved with three vertices of weight 2 and one vertex of weight 1).

A different notion of perfection in Roman domination was considered in [7]. In that paper, the authors study Roman dominating functions in which the vertices of weight 1 and 2 induce an independent set. Another related variant of Roman domination in which each vertex of weight 0 must be adjacent to at least two vertices weighted 2 or one vertex weighted 3 is explored in [1]; the vertices with weight 1 must also be adjacent to at least one vertex with weight 2 or 3, though it is shown there that no weight 1 vertices are ever needed.

In this paper, we show that if G is a tree on $n \ge 3$ vertices, then $\gamma_R^p(G) \le \frac{4}{5}n$, and we characterize the trees achieving equality in this bound.

2. Notation

For a subset S of vertices of a graph G, the subgraph induced by S is denoted by $G[S]$. The subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$.

The *distance* between two vertices u and v is the length of a shortest (u, v)-path in G. The *eccentricity* of a graph G is the maximum distance between any two vertices in G.

A *leaf* is a vertex of degree 1, while its neighbor is a *support vertex*. A *star* is the graph $K_{1,k}$, where $k \ge 1$. For a star with $k > 1$ leaves, the *central vertex* is the unique vertex of degree greater than one. For $r, s \ge 1$, a *double star* $S(r, s)$ is the tree with exactly two vertices that are not leaves, one of which has r leaf neighbors and the other s leaf neighbors. We denote a path on n vertices by P_n.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \ne r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The set of children of v is denoted by $C(v)$. A *descendant* of v is a vertex $u \ne v$ such that the unique (r, u)-path contains v, while an *ancestor* of v is a vertex $u \ne v$ that belongs to the (r, v)-path in T. In particular, every child of v is a descendant of v while the parent of v is an ancestor of v. The *grandparent* of v is the ancestor of v at distance 2 from v. A *grandchild* of v is the descendant of v at distance 2 from v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by diam(G). The *center* of a graph G is the set of all vertices of minimum eccentricity, and a *central* vertex of G is a vertex that belongs to its center. That is, a central vertex of G is a vertex with eccentricity equal to its radius. In particular, a path P_n has a unique central vertex if n is odd, and has two (adjacent) central vertices if n is even.

As a shorthand, we shall use the standard notation $[k] = \{1, \ldots, k\}$.

3. Main result

Let $\mathcal{T}$ be the family of all trees T whose vertex set can be partitioned into sets, each set inducing a path P_5 on five vertices, such that the subgraph induced by the central vertices of these P_5's is connected. We call the subtree induced by these central vertices the *underlying subtree* of the resulting tree T, and we call each such path P_5 a *base path* of the tree T. A tree in the family $\mathcal{T}$ with six base paths and whose underlying subtree is a path P_6 is illustrated in Fig. 1.

We shall prove the following result.

Theorem 1. *If T is a tree of order $n \ge 3$, then $\gamma_R^p(T) \le \frac{4}{5}n$, with equality if and only if $T \in \mathcal{T}$.*

As an immediate corollary of Theorem 1, we have the following result due to Chambers, Kinnersley, Prince and West [2].

Corollary 1 ([2]). *If T is a tree of order $n \ge 3$, then $\gamma_R(T) \le \frac{4}{5}n$, with equality if and only if $T \in \mathcal{T}$.*

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