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Total domination stability in graphs

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ABSTRACT

A set D of vertices in an isolate-free graph G is a total dominating set of G if every vertex is adjacent to a vertex in D. The total domination number, $\gamma_t(G)$, of G is the minimum cardinality of a total dominating set of G. We note that $\gamma_t(G) \ge 2$ for every isolate-free graph G. A non-isolating set of vertices in G is a set of vertices whose removal from G produces an isolate-free graph. The γ_t^--stability, denoted $\mathrm{st}^-_{\gamma_t}(G)$, of G is the minimum size of a non-isolating set S of vertices in G whose removal decreases the total domination number. We show that if G is a connected graph with maximum degree Δ satisfying $\gamma_t(G) \ge 3$, then $\mathrm{st}^-_{\gamma_t}(G) \le 2\Delta - 1$, and we characterize the infinite family of trees that achieve equality in this upper bound. The total domination stability, $\mathrm{st}_{\gamma_t}(G)$, of G is the minimum size of a non-isolating set of vertices in G whose removal changes the total domination number. We prove that if G is a connected graph with maximum degree Δ satisfying $\gamma_t(G) \ge 3$, then $\mathrm{st}_{\gamma_t}(G) \le 2\Delta - 1$.

1. Introduction

The concept of domination stability in graphs was introduced in 1983 by Bauer, Harary, Nieminen and Suffel [1] and has been studied, for example, in [13]. We introduce and study the total version of domination stability. We demonstrate that these two versions differ significantly.

A *dominating set* of a graph G with vertex set $V(G)$ is a set D of vertices of G such that every vertex in $V(G) \setminus D$ is adjacent to a vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The concept of domination critical graphs is well studied in the literature (see, for example, [1–4,6,8,9,15,16]). We focus on domination stability in graphs. As defined in [1], the γ^--*stability* of G, denoted by $\gamma^-(G)$, is the minimum number of vertices whose removal decreases the domination number, and the γ^+-*stability* of G, denoted by $\gamma^+(G)$, is the minimum number of vertices whose removal increases the domination number. The domination stability of G, denoted $\mathrm{st}_\gamma(G)$, is the minimum number of vertices whose removal changes the domination number. Thus, $\mathrm{st}_\gamma(G) = \min\{\gamma^-(G), \gamma^+(G)\}$.

An *isolate-free graph* is a graph with no isolated vertex. A *total dominating set*, abbreviated TD-set, of an isolate-free graph G is a set D of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in D. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A *non-isolating set* of vertices in G is a set D of vertices such that $G - D$ is an isolate-free graph, where $G - D$ denotes the graph obtained from G by removing D and all edges incident with vertices in D. Let $\mathrm{NI}(G)$ denote the set of all non-isolating sets of vertices of G. The concept of total domination critical graphs is well studied in the literature (see, for example, [5,7,10,11,17].) Chapter 11 in the book [12] is devoted to total domination critical graphs.

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Unless otherwise stated, let G be an isolate-free graph. The γ_t^--stability of G, denoted $\mathrm{st}^-_{\gamma_t}(G)$, is the minimum size of a non-isolating set S of vertices in G whose removal decreases the total domination number. Thus,

$$\mathrm{st}^-_{\gamma_t}(G) = \min_{S\in \mathrm{NI}(G)} \{ |S| : \gamma_t(G-S) < \gamma_t(G)\}.$$

The γ_t^+-stability of G, denoted $\mathrm{st}^+_{\gamma_t}(G)$, is the minimum size of a non-isolating set of vertices in G whose removal increases the total domination number, if such a set exists. In this case,

$$\mathrm{st}^+_{\gamma_t}(G) = \min_{S\in \mathrm{NI}(G)} \{ |S| : \gamma_t(G-S) > \gamma_t(G)\}.$$

If no such non-isolating set exists whose removal increases the total domination number, we define $\mathrm{st}^+_{\gamma_t}(G) = \infty$. As a trivial example, $\mathrm{st}^-_{\gamma_t}(P_7) = 2$ while $\mathrm{st}^+_{\gamma_t}(P_7) = \infty$.

The *total domination stability* of G, denoted $\mathrm{st}_{\gamma_t}(G)$, is the minimum size of a non-isolating set S of vertices in G whose removal changes the total domination number. Thus,

$$\mathrm{st}_{\gamma_t}(G) = \min_{S\in \mathrm{NI}(G)} \{ |S| : \gamma_t(G-S) \neq \gamma_t(G)\} = \min\{\mathrm{st}^-_{\gamma_t}(G), \mathrm{st}^+_{\gamma_t}(G)\}.$$

2. Preliminaries

For notation and graph theory terminology we generally follow [12]. The *order* of G is denoted by $n(G) = |V(G)|$, and the *size* of G by $m(G) = |E(G)|$. We denote the *degree* of a vertex v in the graph G by $d_G(v)$. A vertex of degree 0 is called an *isolated vertex*. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). The *open neighborhood* of v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we simply write $d(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ instead of $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively.

For a subset S of vertices of G, the subgraph induced by S is denoted by $G[S]$. The subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G-S$. The set S is an *open packing* if the open neighborhoods of vertices in S are pairwise disjoint. The *open packing number*, $\rho^0(G)$, of G is the maximum cardinality of an open packing in G.

A *non-trivial graph* is a graph of order at least 1. A path and cycle on n vertices are denoted by P_n and C_n, respectively. A complete graph on n vertices is denoted by K_n, while a complete bipartite graph with partite sets of size ℓ and m is denoted by $K_{\ell,m}$. A *star* is the graph $K_{1,k}$, where $k \geq 1$. For $r, s \geq 1$, a *double star* $S(r, s)$ is the tree with exactly two vertices that are not leaves, one of which has r leaf-neighbors and the other s leaf-neighbors.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The set of children of v is denoted by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v, while an *ancestor* of v is a vertex $u \neq v$ that belongs to the (r, v)-path in T. In particular, every child of v is a descendant of v while the parent of v is an ancestor of v.

The *distance* between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. We use the standard notation $[k] = \{1, \dots, k\}$.

Following the original paper of Bauer et al. [1], we consider the *null graph* K_0 (also called the *order-zero graph*), which is the unique graph having no vertices and hence has order zero, as a graph. With this consideration, the domination stability of a non-trivial graph is always defined. In particular, $\mathrm{st}_\gamma(K_n) = n$ since $\gamma(K_n) = 1$ and removing all vertices from the complete graph K_n on n vertices produces the null graph with domination number zero. Bauer et al. [1] established the following fundamental upper bound on the domination stability of a graph.

Theorem 1 ([1]). *For every nontrivial graph G, we have* $\mathrm{st}_\gamma(G) \leq \delta(G) + 1$.

An immediate consequence of Theorem 1, we have the following result observed by Rad et al. [13].

Observation 2 ([13]). *If $G \ncong K_n$ is a graph of order n, then* $\mathrm{st}_\gamma(G) \leq n - 1$.

Considering the null graph K_0 as a graph, we have the following trivial observation.

Observation 3. *If G is a graph of order n and $\gamma_t(G) = 2$, then $\mathrm{st}^-_{\gamma_t}(G) = n$.*

In view of Observation 3, it is only of interest for us to consider isolate-free graphs G with $\gamma_t(G) \geq 3$ when determining $\mathrm{st}^-_{\gamma_t}(G)$. With this assumption, we note that if u and v are adjacent vertices in G, then the set $S = V(G)\setminus\{u, v\}$ is a non-isolating set of vertices in G and $\gamma_t(G-S) = \gamma_t(K_2) = 2 < \gamma_t(G)$. Thus, the γ_t^--stability of an isolate-free graph G with $\gamma_t(G) \geq 3$ is at most two less than its order. We state this formally as follows.

Observation 4. *If G is a graph of order n and $\gamma_t(G) \geq 3$, then $\mathrm{st}^-_{\gamma_t}(G) \leq n - 2$.*