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Characterizing minimally flat symmetric hypergraphs

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ABSTRACT

In Kaszanitzky and Schulze (2017) we gave necessary conditions for a symmetric d-picture (i.e., a symmetric realization of an incidence structure in $\mathbb{R}^d$) to be minimally flat, that is, to be non-liftable to a polyhedral scene without having redundant constraints. These conditions imply very simply stated restrictions on the number of those structural components of the picture that are fixed by the elements of its symmetry group. In this paper we show that these conditions on the fixed structural components, together with the standard non-symmetric counts, are also sufficient for a plane picture which is generic with three-fold rotational symmetry $\mathcal{C}_3$ to be minimally flat. This combinatorial characterization of minimally flat $\mathcal{C}_3$-generic pictures is obtained via a new inductive construction scheme for symmetric sparse hypergraphs. We also give a sufficient condition for sharpness of pictures with $\mathcal{C}_3$ symmetry.

1. Introduction

1.1. Background and motivation

The vertical projection of a spatial polyhedral scene with flat faces yields a straight line drawing of the corresponding incidence structure in the projection plane. Conversely, given an incidence structure S and a straight line drawing of S in the plane, one may ask whether this drawing can be 'lifted' to a polyhedral scene, i.e., whether it is the vertical projection of a spatial polyhedral scene. This is a well studied question in Discrete Geometry which has some beautiful connections to areas such as Geometric Rigidity Theory and Polytope Theory [6–10]. Moreover, this problem has important applications in Artificial Intelligence, Computer Vision and Robotics [4,5].

A fundamental result in Scene Analysis is Whiteley's combinatorial characterization of all incidence structures which are 'minimally flat' if realized generically in the plane, where a realization of an incidence structure is called minimally flat if it is non-liftable to a spatial polyhedral scene, but the removal of any incidence yields a liftable structure. This characterization was conjectured by Sugihara in 1984 [3] and proved by Whiteley in 1989 [7], and it is given in terms of sparsity counts on the number of vertices, faces and incidences of the given incidence structure.

Since symmetry is ubiquitous in both man-made structures and structures found in nature, it is natural to consider the impact of symmetry on the liftability properties of straight line drawings of incidence structures. Recently, we used methods from group representation theory to derive additional necessary conditions for a symmetric realization of an incidence structure to be minimally flat [2]. These conditions can be formulated in a very simple way in terms of the numbers of vertices, faces and incidences that are fixed under the various symmetries of the structure. We conjectured in [2] that these added conditions, together with the standard Sugihara–Whiteley counts are also sufficient for a symmetric incidence structure to be minimally flat, provided that it is realized generically with the given symmetry group.

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In this paper we verify this conjecture for the symmetry group $\mathcal{C}_3$ which is generated by a three-fold rotation (i.e., a rotation by 120°) in the plane. This result is obtained via a new symmetry-adapted recursive construction for symmetric sparse hypergraphs. Moreover, we give a sufficient condition for $\mathcal{C}_3$-symmetric generic incidence structures to lift to a sharp polyhedral scene (i.e., a scene where each pair of faces sharing a vertex lie in separate planes). Finally, we provide some observations regarding extensions of these results to other symmetry groups in the plane.

1.2. Basic definitions

A *(polyhedral) incidence structure* S is an abstract set of vertices V, an abstract set of faces F, and a set of incidences $I \subseteq V \times F$.

A $(d-1)$-*picture* is an incidence structure S together with a corresponding location map $r : V \to \mathbb{R}^{d-1}$, $r_i = (x_i, y_i, \dots, w_i)^T$, and is denoted by $S(r)$.

A d-*scene* $S(p, P)$ is an incidence structure $S = (V, F; I)$ together with a pair of location maps, $p : V \to \mathbb{R}^d$, $p_i = (x_i, \dots, w_i, z_i)^T$, and $P : F \to \mathbb{R}^d$, $P^j = (A^j \dots, C^j, D^j)^T$, such that for each $(i, j) \in I$ we have $A^j x_i + \cdots + C^j w_i + z_i + D^j = 0$. (We assume that no hyperplane is vertical, i.e., is parallel to the vector $(0, \dots, 0, 1)^T$.)

A *lifting* of a $(d-1)$-picture $S(r)$ is a d-scene $S(p, P)$, with the vertical projection $\Pi(p) = r$. That is, if $p_i = (x_i, \dots, w_i, z_i)^T$, then $r_i = (x_i, \dots, w_i)^T = \Pi(p_i)$.

A lifting $S(p, P)$ is *trivial* if all the faces lie in the same plane. Further, $S(p, P)$ is *folded* (or *non-trivial*) if some pair of faces have different planes, and is *sharp* if each pair of faces sharing a vertex have distinct planes. A picture is called *sharp* if it has a sharp lifting. Moreover, a picture which has no non-trivial lifting is called *flat* (or *trivial*). A picture with a non-trivial lifting is called *foldable*.

The *lifting matrix* for a picture $S(r)$ is the $|I| \times (|V| + d|F|)$ coefficient matrix $M(S, r)$ of the system of equations for liftings of a picture $S(r)$: For each $(i, j) \in I$, we have the equation $A^j x_i + B^j y_i + \cdots + C^j w_i + z_i + D^j = 0$, where the variables are ordered as $[\dots, z_i, \dots; \dots, A^j, B^j, \dots, D^j, \dots]$. That is the row corresponding to $(i, j) \in I$ is:

$$(i,j)\ \Big[\underbrace{0\dots0 \overset{i}{1} \ 0\dots0}_{|V|} \ \Big\| \ \underbrace{0\dots0 \ \overbrace{r_i \ \ 1}^{j} \ 0\dots0}_{d|F|}\Big]$$

A $(d-1)$-picture $S(r)$ is called *generic* if for every $r' : V \to \mathbb{R}^{d-1}$, the rank of every square submatrix of the lifting matrix $M(S, r)$ is greater than or equal to the rank of the corresponding submatrix of $M(S, r')$. So in particular, $M(S, r)$ has maximum rank among all lifting matrices $M(S, r')$.

Theorem 1.1 (*Picture Theorem*). [7,9] *A generic* $(d-1)$-*picture of an incidence structure* $S = (V, F; I)$ *with at least two faces has a sharp lifting, unique up to lifting equivalence, if and only if* $|I| = |V| + d|F| - (d+1)$ *and* $|I'| \le |V'| + d|F'| - (d+1)$ *for all subsets* I' *of incidences inducing the vertex set* $V' \subseteq V$ *and face set* $F' \subseteq F$ *with* $|F'| \ge 2$.

A generic picture of S *has independent rows in the lifting matrix if and only if for all non-empty subsets* I' *of incidences, we have* $|I'| \le |V'| + d|F'| - d$.

It follows from the Picture Theorem that a generic $(d-1)$-picture of an incidence structure $S = (V, F; I)$ is flat with independent rows in the lifting matrix if and only if $|I| = |V| + d|F| - d$ and $|I'| \le |V'| + d|F'| - d$ for all non-empty subsets I' of incidences. The removal of any incidence from such a picture results in a foldable picture, and hence we call such a picture *minimally flat*. We may think of the minimally flat generic pictures $S(r)$ as the bases of the row matroid of the lifting matrix $M(\tilde{S}, r)$ of a flat picture $(\tilde{S}, r)$ on an incidence structure $\tilde{S} = (V, F; \tilde{I})$ with $I \subseteq \tilde{I}$. Therefore, the Picture Theorem may be considered as an analog of the celebrated Laman's Theorem in Geometric Rigidity Theory, which gives a description of the bases of the generic rigidity matroid of a rigid graph in dimension 2 [9,10].

1.3. Symmetric incidence structures and pictures

An *automorphism* of an incidence structure $S = (V, F; I)$ is a pair $\alpha = (\pi, \sigma)$, where π is a permutation of V and σ is a permutation of F such that $(v, f) \in I$ if and only if $(\pi(v), \sigma(f)) \in I$ for all $v \in V$ and $f \in F$. For simplicity, we will write $\alpha(v)$ for $\pi(v)$ and $\alpha(f)$ for $\sigma(f)$.

The automorphisms of S form a group under composition, denoted Aut(S). An *action* of a group Γ on S is a group homomorphism $\theta : \Gamma \to \text{Aut}(S)$. The incidence structure S is called Γ-*symmetric* (with respect to θ) if there is such an action. For simplicity, if θ is clear from the context, we will sometimes denote the automorphism $\theta(\gamma)$ simply by γ.

Let Γ be an abstract group, and let S be a Γ-symmetric incidence structure (with respect to θ). Further, suppose there exists a group representation $\tau : \Gamma \to O(\mathbb{R}^{d-1})$. Then we say that a picture $S(r)$ is Γ-*symmetric* (with respect to θ and τ) if

$$\tau(\gamma)(r_i) = r_{\theta(\gamma)(i)} \text{ for all } i \in V \text{ and all } \gamma \in \Gamma. \tag{1}$$

In this case we also say that $\tau(\Gamma) = \{\tau(\gamma) | \gamma \in \Gamma\}$ is a *symmetry group* of $S(r)$, and each element of $\tau(\Gamma)$ is called a *symmetry operation* of $S(r)$.