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Strong resolving graphs: The realization and the characterization problems

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A B S T R A C T

The strong resolving graph G_{SR} of a connected graph G was introduced in Oellermann and Peters-Fransen (2007) as a tool to study the strong metric dimension of G. Basically, it was shown that the problem of finding the strong metric dimension of G can be transformed to the problem of finding the vertex cover number of G_{SR}. Since then, several articles on the strong metric dimension of graphs which are using this tool have been published. However, the tool itself has remained unnoticed as a properly structure. In this paper, we survey the state of knowledge on the strong resolving graphs, and also derive some new results regarding its properties.

1. Introduction

Graphs are basic combinatorial structures, and transformations of structures are fundamental to the development of mathematics. Particularly, in graph theory, some elementary transformations generate a new graph from an original one by some simple local changes, such as addition or deletion of a vertex or of an edge, merging and splitting of vertices, edge contraction, etc. Other advanced transformations create a new graph from the original one by complex changes, such as complement graph, line graph, total graph, graph power, dual graph, strong resolving graph, etc.

Some of these transformations of graphs emerged as a natural tool to solve practical problems. In other cases, the problem of finding a specific parameter of a graph has become the problem of finding another parameter of another graph obtained from the original one. This is the case of the strong resolving graph G_{SR} of a connected graph G which was introduced in [33] as a tool to study the strong metric dimension of G. Basically, it was shown that the problem of finding the strong metric dimension of G can be transformed to the problem of finding the vertex cover number of G_{SR}. Since then, several articles dealing with the strong resolving graph have been published. However, in almost all these works the results related to the strong resolving graph are not explicit, as they implicitly appear as a part of the proofs of main results concerning the strong metric dimension. In this sense, this interesting construction has passed in front of researchers's eyes without the attention that should require. In this paper, we would like to motivate the graph theory community to have a deeper look into this graph transformation. Accordingly, herein we survey the state of knowledge on the strong resolving graph and also derive some new results.

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For a graph transformation, there are two general problems [11], which we shall formulate in terms of strong resolving graphs:

- **Realization Problem.**[1] Determine which graphs have a given graph as their strong resolving graphs.
- **Characterization Problem.** Characterize those graphs that are strong resolving graphs of some graphs.

The majority of results presented in this paper concerns the above mentioned problems. Basically, we focus on the following graph equation

$$G_{SR} \cong H, \quad (1)$$

i.e., the goal is to find all pairs of graphs G and H satisfying (1).

The remainder of the paper is structured as follows. Section 1.1 covers general notation and terminology. Section 1.2 is devoted to introduce the strong metric dimension, whereas Section 1.3 introduces the strong resolving graph. In Section 2 we study the realization problem for some specific families of graphs, while in Section 3 we collect the known results related to the characterization problem of product graphs. We close our exposition with a collection of open problems to be dealt with.

1.1. Notation and terminology

We continue by establishing the basic terminology and notations which is used throughout this work. For the sake of completeness we refer the reader to the books [6,12,38]. Graphs considered herein are undirected, finite and contain neither loops nor multiple edges. Let G be a graph of order $n = |V(G)|$. A graph is nontrivial if $n \geq 2$. We use the notation $u \sim v$ for two adjacent vertices u and v of G. For a vertex v of G, $N_G(v)$ denotes the set of neighbors that v has in G, *i.e.*, $N_G(v) = \{u \in V(G) : u \sim v\}$. The set $N_G(v)$ is called the *open neighborhood of a vertex* v in G and $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood of a vertex* v in G. The *degree* of a vertex v of G is denoted by $\delta_G(v)$, *i.e.*, $\delta_G(v) = |N_G(v)|$. The *open neighborhood of a set* S of vertices of G is $N_G(S) = \bigcup_{v\in S} N_G(v)$ and the *closed neighborhood of* S is $N_G[S] = N_G(S) \cup S$.

We use the notation K_n, C_n, P_n, and N_n for the *complete graph*, *cycle*, *path*, and *empty graph*, respectively. Moreover, we write $K_{s,t}$ for the *complete bipartite graph* of order $s + t$ and in particular $K_{1,n}$ for the *star* of order $n + 1$. A vertex of degree one in a tree T is called a *leaf* and the number of leaves in T is denoted by $l(T)$.

The *distance* between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G. The *diameter*, $D(G)$, of G is the largest distance between any two vertices of G and two vertices $u, v \in V(G)$ such that $d_G(u, v) = D(G)$ are called *diametral*. If G is not connected, then we assume that the distance between any two vertices belonging to different components of G is infinity and, thus, its diameter is $D(G) = \infty$. A graph G is 2-*antipodal* if for each vertex $x \in V(G)$ there exists exactly one vertex $y \in V(G)$ such that $d_G(x, y) = D(G)$. For instance, even cycles and hypercubes are 2-antipodal graphs.

We recall that the *complement* of G is the graph G^c with the same vertex set as G and $uv \in E(G^c)$ if and only if $uv \notin E(G)$. The *subgraph induced by a set* X is denoted by $\langle X\rangle$. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbors is a complete graph. Given a graph G, we denote by $\sigma(G)$ the set of simplicial vertices of G.

A *clique* in G is a set of pairwise adjacent vertices. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique in G. Two distinct vertices u, v are called *true twins* if $N_G[u] = N_G[v]$. In this sense, a vertex x is a *twin* if there exists $y \neq x$ such that they are true twins. We say that $X \subset V(G)$ is a *twin-free clique* in G if the subgraph induced by X is a clique and for every $u, v \in X$ it follows $N_G[u] \neq N_G[v]$, *i.e.*, the subgraph induced by X is a clique and it contains no true twins. The *twin-free clique number* of G, denoted by $\varpi(G)$, is the maximum cardinality among all twin-free cliques in G. So, $\omega(G) \geq \varpi(G)$. We refer to a $\varpi(G)$-set in a graph G as a twin-free clique of cardinality $\varpi(G)$. Fig. 1 shows examples of basic concepts such as true twins and twin-free clique.

For the remainder of the paper, definitions will be introduced whenever a concept is needed.

1.2. Strong metric dimension of graphs

A vertex $w \in V(G)$ *strongly resolves* two different vertices $u, v \in V(G)$ if $d_G(w, u) = d_G(w, v) + d_G(v, u)$ or $d_G(w, v) = d_G(w, u) + d_G(u, v)$, *i.e.*, there exists some shortest $w - u$ path containing v or some shortest $w - v$ path containing u. A set S of vertices in a connected graph G is a *strong metric generator* for G if every two vertices of G are strongly resolved by some vertex in S. The minimum cardinality among all strong metric generators for G is called the *strong metric dimension* and is denoted by $dim_s(G)$. A *strong metric basis* of G is a strong metric generator for G of cardinality $dim_s(G)$.

Several researches on the strong metric dimension of graphs have recently been developed. For instance, the trivial bounds $1 \leq dim_s(G) \leq n - 1$ are known from the first works as well as characterizations on whether they are achieved. Moreover, it has been noticed that the strong metric dimension of several graphs can be straightforwardly computed for some basic examples which we next remark.

[1] This problem was called Determination Problem in [11].