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Total rainbow connection of digraphs

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ABSTRACT

An edge-coloured path is *rainbow* if its edges have distinct colours. For a connected graph G, the *rainbow connection number* (resp. *strong rainbow connection number*) of G is the minimum number of colours required to colour the edges of G so that any two vertices of G are connected by a rainbow path (resp. rainbow geodesic). These two graph parameters were introduced by Chartrand, Johns, McKeon, and Zhang in 2008. Krivelevich and Yuster generalised this concept to the vertex-coloured setting. Similarly, Liu, Mestre, and Sousa introduced the version which involves total-colourings.

Dorbec, Schiermeyer, Sidorowicz, and Sopena extended the concept of the rainbow connection to digraphs. In this paper, we consider the (strong) total rainbow connection number of digraphs. Results on the (strong) total rainbow connection number of biorientations of graphs, tournaments, and cactus digraphs are presented.

1. Introduction

All graphs and digraphs considered in this paper are finite and simple. That is, we do not allow the existence of loops, multiple edges (for graphs), and multiple directed arcs (for digraphs). We follow the terminology and notation of Bollobás [3] for those not defined here.

The concept of rainbow connection in graphs was introduced by Chartrand, Johns, McKeon, and Zhang [4]. An edge-coloured path is *rainbow* if its edges have distinct colours. An edge-colouring of a connected graph G is *rainbow connected* if any two vertices of G are connected by a rainbow path. The *rainbow connection number* of G, denoted by $rc(G)$, is the minimum number of colours in a rainbow connected edge-colouring of G. An edge-colouring of G is *strongly rainbow connected* if for every pair of vertices u and v, there exists a rainbow $u-v$ geodesic, i.e., a path of length equal to the distance between u and v. The minimum number of colours in a strongly rainbow connected edge-colouring of G is the *strong rainbow connection number* of G, denoted by $src(G)$.

As a natural counterpart to the rainbow connection of edge-coloured graphs, Krivelevich and Yuster [8]; and Li, Mao, and Shi [10], proposed the concept of (strong) rainbow vertex-connection. A vertex-coloured path is *vertex-rainbow* if its internal vertices have distinct colours. A vertex-colouring of a connected graph G is *rainbow vertex-connected* (resp. *strongly rainbow vertex-connected*) if any two vertices of G are connected by a vertex-rainbow path (resp. geodesic). The *rainbow vertex-connection number* of G, denoted by $rvc(G)$, is the minimum number of colours in a rainbow vertex-connected vertex-colouring of G. The *strong rainbow vertex-connection number* of G, denoted by $srvc(G)$, is the minimum number of colours in

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a strongly rainbow vertex-connected vertex-colouring of G. We refer the reader to the survey [11] and the monograph [12] on the subject of rainbow connection in graphs.

Liu, Mestre, and Sousa [13]; and Chen, Li, Liu, and Liu [5], proposed the concept of (strong) total rainbow connection. A total-coloured path is *total-rainbow* if its edges and internal vertices have distinct colours. A total-colouring of a connected graph G is *total rainbow connected* (resp. *strongly total rainbow connected*) if any two vertices are connected by a total-rainbow path (resp. geodesic). The *total rainbow connection number* of G, denoted by $trc(G)$, is the minimum number of colours in a total rainbow connected total-colouring of G. The *strong total rainbow connection number* of G, denoted by $strc(G)$, is the minimum number of colours in a strongly total rainbow connected total-colouring of G.

In [6], Dorbec, Schiermeyer, Sidorowicz, and Sopena introduced the concept of rainbow connection of digraphs. A *directed path*, or simply a *path* P, is a digraph consisting of a sequence of vertices $v_0, v_1, \ldots, v_\ell$ and arcs $v_{i-1}v_i$ for $1 \le i \le \ell$. We also say that P is a $v_0 - v_\ell$ *path*, and its *length* is the number of arcs ℓ. A digraph D is *strongly connected* if for any ordered pair of vertices (u, v) in D, there exists a $u - v$ path. An arc-coloured path is *rainbow* if its arcs have distinct colours. Let D be a strongly connected digraph. An arc-colouring of D is *rainbow connected* if for any ordered pair of vertices (u, v) in D, there is a rainbow $u - v$ path. The *rainbow connection number* of D, denoted by $\overrightarrow{rc}(D)$, is the minimum number of colours in a rainbow connected arc-colouring of D. Alva-Samos and Montellano-Ballesteros [1] then introduced the notion of strong rainbow connection of digraphs. An arc-colouring of D is *strongly rainbow connected* if for any ordered pair of vertices (u, v), there is a rainbow $u - v$ geodesic, i.e., a rainbow $u - v$ path of minimum length. The *strong rainbow connection number* of D, denoted by $\overrightarrow{src}(D)$, is the smallest possible number of colours in a strongly rainbow connected arc-colouring of D. We have $\mathrm{diam}(D) \le \overrightarrow{rc}(D) \le \overrightarrow{src}(D)$, where $\mathrm{diam}(D)$ denotes the diameter of D. Subsequently, there have been some results on this topic, which considered many different classes of digraphs [2,7,14]. Very recently, Lei, Li, Liu, and Shi [9] introduced the (strong) rainbow vertex-connection of digraphs. A vertex-coloured directed path is *vertex-rainbow* if its internal vertices have distinct colours. A vertex-colouring of D is *rainbow vertex-connected* (resp. *strongly rainbow vertex-connected*) if for any ordered pair of vertices (u, v) in D, there exists a vertex-rainbow $u - v$ path (resp. geodesic). The *rainbow vertex-connection number* of D, denoted by $\overrightarrow{rvc}(D)$, is the minimum number of colours in a rainbow vertex-connected vertex-colouring of D. The *strong rainbow vertex-connection number* of D, denoted by $\overrightarrow{srvc}(D)$, is the minimum number of colours in a strongly rainbow vertex-connected vertex-colouring of D. We have $\mathrm{diam}(D) - 1 \le \overrightarrow{rvc}(D) \le \overrightarrow{srvc}(D)$.

In this paper, we introduce the concept of total rainbow connection of digraphs. Let D be a strongly connected digraph. A total-coloured directed path is *total-rainbow* if its arcs and internal vertices have distinct colours. A total-colouring of D is *total rainbow connected* if for any ordered pair of vertices (u, v) in D, there exists a total-rainbow $u - v$ path. The *total rainbow connection number* D, denoted by $\overrightarrow{trc}(D)$, is the minimum number of colours in a total rainbow connected total-colouring of D. Likewise, a total-colouring of D is *strongly total rainbow connected* if for any ordered pair of vertices (u, v), there exists a total-rainbow $u - v$ geodesic. The *strong total rainbow connection number* of D, denoted by $\overrightarrow{strc}(D)$, is the minimum number of colours in a strongly total rainbow connected total-colouring of D.

This paper is organised as follows. In Section 2, we present several general results about the parameters $\overrightarrow{trc}(D)$ and $\overrightarrow{strc}(D)$, as well as their relationships to the parameters $\overrightarrow{rc}(D)$, $\overrightarrow{src}(D)$, $\overrightarrow{rvc}(D)$, and $\overrightarrow{srvc}(D)$. In Section 3, we compute the parameters $\overrightarrow{trc}(D)$ and $\overrightarrow{strc}(D)$ for some specific digraphs D. In Section 4, we study the parameters $\overrightarrow{trc}(T)$ and $\overrightarrow{strc}(T)$ for tournaments T. Finally in Section 5, we consider the parameters $\overrightarrow{rvc}(Q)$ and $\overrightarrow{trc}(Q)$ for cactus digraphs Q.

2. Definitions, remarks, and results for general digraphs

We begin with some definitions about digraphs. For a digraph D, its vertex and arc sets are denoted by $V(D)$ and $A(D)$. For an arc $uv \in A(D)$, we say that v is an *out-neighbour* of u, and u is an *in-neighbour* of v. Moreover, we call uv an *in-arc* of v and an *out-arc* of u. We denote the set of *out-neighbours* (resp. *in-neighbours*) of u in D by $\Gamma^+(u)$ (resp. $\Gamma^-(u)$). Let $\Gamma[u] = \Gamma^+(u) \cup \Gamma^-(u) \cup \{u\}$. For a strongly connected digraph D, and $u, v \in V(D)$, the distance from u to v (i.e., the length of a shortest $u - v$ path) in D is denoted by $d(u, v)$, or $d_D(u, v)$ if we wish to emphasise that the distance is taken in the digraph D. Let $\mathrm{diam}(D)$ denote the diameter of D.

If $uv, vu \in A(D)$, then we say that uv and vu are *symmetric arcs*. If $uv \in A(D)$ and $vu \notin A(D)$, then uv is an *asymmetric arc*. The digraph D is an *oriented graph* if every arc of D is asymmetric. A *tournament* is an oriented graph where every two vertices have one asymmetric arc joining them. A *cactus* is a strongly connected oriented graph where each arc belongs to exactly one directed cycle. Given a graph G, its *biorientation* is the digraph $\overleftrightarrow{G}$ obtained by replacing each edge uv of G by the pair of symmetric arcs uv and vu. Let $\overrightarrow{P}_n$ and $\overrightarrow{C}_n$ denote the directed path and directed cycle of order n, respectively (where $n \ge 3$ for $\overrightarrow{C}_n$), i.e., we may let $V(\overrightarrow{P}_n) = V(\overrightarrow{C}_n) = \{v_0, \ldots, v_{n-1}\}$, and $A(\overrightarrow{P}_n) = \{v_0v_1, v_1v_2 \ldots, v_{n-2}v_{n-1}\}$ and $A(\overrightarrow{C}_n) = A(\overrightarrow{P}_n) \cup v_{n-1}v_0$. If C is a directed cycle and $u, v \in V(C)$, we write uCv for the unique $u - v$ directed path in C.

For a subset $X \subset V(D)$, we denote by $D[X]$ the subdigraph of D induced by X. Given two digraphs D and H, and $u \in V(D)$, we define $D_{u \to H}$ to be the digraph obtained from D and H by replacing the vertex u by a copy of H, and replacing each arc xu (resp. ux) in D by all the arcs xv (resp. vx) for $v \in V(H)$. We say that $D_{u \to H}$ is *obtained from* D *by expanding* u *into* H. Note that the digraph obtained from D by expanding every vertex into H is also known as the *lexicographic product* $D \circ H$.

Now, we shall present some remarks and basic results for the total rainbow connection and strong total rainbow connection numbers, for general digraphs and biorientations of graphs. We first note that in a total rainbow connected