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Multi-player Small Nim with Passes

Wen An Liu *, Jing Jing Zhou

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, People's Republic of China

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A B S T R A C T

In Guy and Nowakowski's *Unsolved Problems in Combinatorial Games*, the following entry is found: "David Gale would like to see an analysis of Nim played with the option of a single pass by either of the players, which may be made at any time up to the penultimate move. It may not be made at the end of the game. Once a player has passed, the game is as in ordinary Nim. The game ends when all heaps have vanished."

This paper investigates the n-person combinatorial game of "Small Nim with Passes", a variant of Nim, where players must always remove objects from the smallest nonempty pile and are allowed to "pass" their turn for a finite number of times. Let N be the number of piles in the game. When the number of players is greater than $N+1$, we determine all game values for all possible positions. The game values are determined completely when the number of players is equal to $N+1$. We also analyze certain cases of positions when the number of players is smaller than $N+1$, and leave some open problems that could be of interest to future research.

1. Introduction

We assume that the reader has some knowledge in combinatorial game theory. Basic definitions can be found in [2,6]. In a 2-person perfect information game two players alternately move until one of them is unable to move at his turn. Among the games of this type are *Nim* [3,11,23], *End-Nim* [10], *Wythoff's game* [5,7–9,19], (s,t)-*Wythoff's game* [1,18,21], *Wythoff-like game* [22], etc.

Naturally it is of interest to generalize as much as possible the theory to n-player games. In 2-player perfect information games, one can always discuss any possible outcome, when each player plays it right i.e. when each player adopts an optimal strategy. But when there are more than two players, this approach does not work out. For instance, it may happen that one player can help any of the other players, while he himself will lose. So the outcome of the game depends on how the group coalitions are formed among the players. In previous literature, several possibilities were investigated.

- *Multi-player without alliance*. See [4,16,24–26].
- *Multi-player with two alliances*. See [12,13,27] and [17, Introduction].
- *Multi-player with alliance system*

Krawec [14] assumed that every player has a fixed set of allegiances to all n players, i.e. an *alliance system* may be defined arbitrarily before the start of a game. Assuming that the chosen alliance system is maintained throughout the game, Krawec provided a method of analyzing n-player impartial games, and derived a recursive function capable of determining which of the n players has a winning strategy.

Krawec [15] developed a method of analyzing n-player impartial combinatorial games where $n-1$ players behave optimally whereas one of the players plays randomly i.e. he makes his moves without any strategy.

* Corresponding author.
E-mail address: liuwenan@126.com (W.A. Liu).

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Liu and Wang ([20], 2017) analyzed "Multi-player subtraction games". Some result given by Krawec in [14] was generalized from $n = 3$ to an arbitrary integer $n \geq 3$, and the order of subtraction games from $k = 2$ to an arbitrary integer $k \geq 2$. The 3-player subtraction games of order 2 were completely analyzed. It turns out that the sequences of game values are always periodic. The explicit representations of pre-periods and periods were presented.

Liu and Duan ([17], 2017) analyzed "Misère N-pile Nim with n players". The game values are completely determined for three cases $n > N+1$, $n = N+1$ and $n = N$.

In the present paper we introduce a class of impartial combinatorial games, *Multi-player Small Nim with Passes*, assuming that the *standard alliance matrix* (to be defined shortly) is adopted.

Definition 1. (i) "Multi-player Small Nim *without* Pass", denoted by $\mathrm{SNim}^{(0)}(N, n)$: There are N piles of counters, and n players who take turns in sequential unchanging order. Each player, at his turn, removes *any* positive integer number of counters from the *smallest* nonempty pile. The first player who cannot make any legal move wins.

(ii) "Multi-player Small Nim *with* s passes", denoted by $\mathrm{SNim}^{(s)}(N, n)$: It is played like $\mathrm{SNim}^{(0)}(N, n)$ with s passes and each pass can be used only once. Once a pass option is used, the game continues in $\mathrm{SNim}^{(s-1)}(N, n)$ i.e. the total number of passes decreases by 1. Once all s passes are used, no further pass option can be used, and the game continues as in $\mathrm{SNim}^{(0)}(N, n)$. A pass option can be used at any time, up to the penultimate move, but cannot be used at the end of the game. The player who cannot make a move wins the game.

A position of $\mathrm{SNim}^{(s)}(N, n)$ can be represented by $[\mathbf{p}; s] = [x_1, x_2, \ldots, x_N; s]$ with $1 \leq x_1 \leq x_2 \leq \cdots \leq x_N$ (where x_i is the number of counters in the ith pile, and s is the total number of passes) as the ordering of the piles makes no difference, and the pile of size 0 can be omitted. For instance, the position $[8, 12, 0, 5, 0, 7; 3]$ can be considered as $[5, 7, 8, 12; 3]$.

The aim of the present paper is to determine the game values $g[\mathbf{p}; s]$ (to be defined shortly, but loosely speaking, the game value $g[\mathbf{p}; s]$ determines the winning player of game $[\mathbf{p}; s]$) for all $\mathbf{p}$, $n \geq 3$ and $s \geq 0$.

If $n > N+1$, the game values $g[\mathbf{p}; s]$ are completely determined in Section 3. Theorem 4 shows that if $n > N+1$, the game value $g[x_1, x_2, \ldots, x_N; s] = N$ for all $s \geq 0$. In particular, this depends on neither the sizes x_i of piles nor the total number s of passes.

If $n = N+1$, the game values $g[\mathbf{p}; s]$ are completely determined in Section 4. Theorem 6 shows that $g[\mathbf{p}; s] = g[\mathbf{p}; \bar{s}]$ for all $s \geq 0$ where $\bar{s} = s \mod n$. Theorem 5 gives all game values $g[\mathbf{p}; \bar{s}]$ by distinguishing $0 \leq \bar{s} \leq n-2$ or $\bar{s} = n-1$.

Section 5 aims to analyze $\mathrm{SNim}^{(s)}(N, n)$ where $3 \leq n \leq N$. The game values are determined for infinitely many triplets (N, n, s). In Section 6, we also leave some open problems that could be of interest to future research.

2. Basic definitions

Throughout the paper, we employ some definitions and notation used in [14,15].

Definition 2. (i) A player shall be referred to P_i where i is an integer in $[0, n-1]$. Unless stated otherwise, player P_0 is the first to move followed by P_1 and so on. After player P_{n-1}, P_0 will play again. Hence all subscripts are taken modulo n.

(ii) Given an n-player game G, the *game value* of G (denoted by $g(G, i)$) is an integer between 0 and $n-1$ (inclusive) which specifies the player, relative to the current player P_i, that can win. For instance, if it is player P_i's turn, and the game value $g(G, i) = j$, then P_{i+j} (with subscript mod n) has a wining strategy.

(iii) Given an n-player game G, by $Opt(G)$ we denote the set of all *options* that the current player can move to by making one legal move. If $Opt(G) = \emptyset$, the empty set, then G is called an *end game* or a *terminal position*.

Definition 3. (i) An *alliance system*, known to all players before the start of the game, is represented by an $n \times n$ matrix of the following form

$$\begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,n-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,n-1} \\ \vdots & \vdots & & \vdots \\ A_{n-1,0} & A_{n-1,1} & \cdots & A_{n-1,n-1} \end{pmatrix}$$

where each entry in the alliance matrix is relative to a particular player i.e. $A_{i,j}$ determines the most preferred player for P_i, relative to that player i. More clearly, given an integer $i \in \{0, 1, 2, \ldots, n-1\}$, the jth preferred player for player P_i would be $P_{i+A_{i,j}}$ i.e. P_i prefers $P_{i+A_{i,0}}$ over $P_{i+A_{i,1}}$ over $P_{i+A_{i,2}}$ ... over $P_{i+A_{i,n-1}}$.

(ii) The following alliance system is called the *Standard Alliance Matrix* (SAM)

$$\begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 0 & 1 & \cdots & n-1 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & n-1 \end{pmatrix}$$