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First order sentences about random graphs: Small number of alternations[☆]

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ABSTRACT

The spectrum of a first order sentence is the set of all α such that $G(n, n^{-\alpha})$ does not obey zero–one law with respect to this sentence. In this paper, we prove that the minimal number of quantifier alternations of a first order sentence with infinite spectrum equals 3. We have also proved that the spectrum of a first-order sentence with quantifier depth 4 has no limit points except possibly the points 1/2 and 3/5.

1. Previous results on zero–one laws

In this paper, we consider first order sentences about graphs (a signature consists of two predicates $\sim$ (adjacency) and $=$ (equality) of arity 2) [15,19]. Recall that a *quantifier depth* $\mathrm{q}(\phi)$ of a sentence ϕ is the minimum number of nested quantifiers required to express the sentence. Let $G(n, p)$ be a binomial random graph [3,7] with n vertices and the probability p of appearing of an edge. We say that $G(n, p)$ *obeys zero–one law w.r.t. a first order sentence* ϕ , if either a.a.s. (asymptotically almost surely) $G(n, p) \models \phi$, or a.a.s. $G(n, p) \models \neg(\phi)$.

Let $S(\phi)$ be the set of all $\alpha > 0$ such that $G(n, n^{-\alpha})$ does not obey zero–one law w.r.t. ϕ. This set is called *a spectrum of* ϕ. In 1988 [11], S. Shelah and J. Spencer proved that there are only rational numbers in $S(\phi)$ for any first order sentence ϕ. In 1990 [12], J. Spencer proved that there exists a first order sentence with an infinite spectrum and the quantifier depth 14. In his paper [14], he also proved that, for a first order sentence ϕ with a quantifier depth k, $S(\phi)\cap(0, 1/(k-1)) = \varnothing$. This result was strengthened by M. Zhukovskii in 2012 [17]: $S(\phi)\cap(0, 1/(k-2)) = \varnothing$. In particular, for any first order sentence ϕ with the quantifier depth 3, $S(\phi)\cap(0, 1) = \varnothing$, and, for any first order sentence ϕ with the quantifier depth 4, $S(\phi)\cap(0, 1/2) = \varnothing$. Later in [8], it was proved that, for any first order sentence ϕ, the set $S(\phi) \cap (1, \infty)$ is finite. In [18], a first order sentence with the quantifier depth 5 and an infinite spectrum was obtained. This formula is given in the statement below.

Theorem 1 (*[18]*). *Let $m \in \mathbb{N}$, $\alpha = \frac{1}{2} + \frac{1}{2(m+1)}$ and $p = n^{-\alpha}$. Then the random graph $G(n, p)$ does not obey zero–one law w.r.t. the sentence*

$$\phi = \exists x_1 \exists x_2 \left[\left(\exists x_3 \exists x_4 \left(\bigwedge_{1\leq i<j\leq 4} (x_i \sim x_j) \right) \right) \wedge (\varphi(x_1, x_2)) \right], \quad (1)$$

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where

$$\varphi(x_1, x_2) = \forall y_1 ([y_1 \sim x_1] \vee [y_1 \sim x_2] \vee [\forall y_2(\neg[(y_2 \sim x_1) \wedge (y_2 \sim y_1)])] \vee$$

$$[\exists z\,(z \sim x_1) \wedge (z \sim x_2) \wedge (\forall u\,[(\neg[(u \sim z) \wedge (u \sim y_1)]) \vee (u \sim x_1) \vee (u \sim x_2)])]).$$

So, a minimal quantifier depth of a first order sentence with an infinite spectrum equals either 4, or 5.

Note that the maximal number of quantifier alternations over all sequences of nested quantifiers in ϕ (Eq. (1)) equals 3 (we call this value the *number of quantifier alternations of* ϕ). It is essential that all the negations are applied to atomic formulas only. A prenex normal form of ϕ with the quantifier depth 8 is given below

$$\tilde{\phi} = \exists x_1 \exists x_2 \exists x_3 \exists x_4 \forall y_1 \forall y_2 \exists z \forall u \left[\left(\bigwedge_{1 \le i < j \le 4} (x_i \sim x_j) \right) \wedge (\tilde{\varphi}(x_1, x_2, y_1, y_2, z, u)) \right], \quad (2)$$

where

$$\tilde{\varphi}(x_1, x_2, y_1, y_2, z, u) = [y_1 \sim x_1] \vee [y_1 \sim x_2] \vee [\neg((y_2 \sim x_1) \wedge (y_2 \sim y_1))] \vee$$

$$[(z \sim x_1) \wedge (z \sim x_2) \wedge (\neg[(u \sim z) \wedge (u \sim y_1)])] \vee [u \sim x_1] \vee [u \sim x_2].$$

This raises the following questions.

1. What is the minimal quantifier depth of a first order sentence with an infinite spectrum, 4 or 5?
2. What is the minimal number of quantifier alternations of a first order sentence with an infinite spectrum, 3 or less?
3. What is the minimal quantifier depth of a first order sentence in a prenex normal form with an infinite spectrum, 4, 5, 6, 7 or 8?

We partially answer these questions in Sections 4 and 5. In Section 3 we state and prove some results on first order formulas, that are used in our answers. Section 2 is devoted to the limit probabilities of properties related to the presence of small subgraphs and extensions in the random graph.

2. Existence and extension statements

Let ϕ be a first order sentence in a prenex normal form. We call ϕ *an existence sentence*, if all quantifiers of ϕ equal $\exists$, i.e. ϕ can be expressed using quantifiers that are all existential. We call ϕ *an extension sentence*, if ϕ can be expressed using a sequence of quantifiers of the form $\forall \dots \forall \exists \dots \exists$. We say that an existence sentence expresses an existence property, and an extension sentence expresses an extension property. The asymptotical behavior of probabilities of the random graph existence and extension properties has been widely studied in [2,6,10,13]. We summarize this study in the result given below.

For an arbitrary graph G denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G correspondingly, denote $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Let G, H be two graphs such that $H \subset G$, $V(H) = \{a_1, \dots, a_s\}$, $V(G) \setminus V(H) = \{b_1, \dots, b_m\}$, $s, m \ge 1$. Let $\rho(H)$ be the maximal value of the fraction $e(Q)/v(Q)$ over all subgraphs $Q \subset H$ ($\rho(H)$ is called *the maximal density* of H). Let $\rho(G, H)$ be the maximal value of the fraction $(e(Q) - e(H))/(v(Q) - v(H))$ over all Q such that $H \subset Q \subset G$. We say that a graph has *the* (G, H)*-extension property*, if, for any its distinct vertices $y_1, \dots, y_s$, there exist distinct vertices $x_1, \dots, x_m$ such that, for all $i \in \{1, \dots, s\}$, $j \in \{1, \dots, m\}$, $y_i \neq x_j$ and the adjacency relation $a_i \sim b_j$ implies the adjacency relation $y_i \sim x_j$.

Theorem 2. *Let $\rho(H) \neq 0$, $p = n^{-\alpha}$. If $\alpha < 1/\rho(H)$, then a.a.s. in $G(n, p)$ there is an induced copy of H. If $\alpha > 1/\rho(H)$, then a.a.s. in $G(n, p)$ there is no copy of H.*

Let $\rho(G, H) \neq 0$, $p = n^{-\alpha}$. If $\alpha < 1/\rho(G, H)$, then a.a.s. $G(n, p)$ has the (G, H)-extension property. If $\alpha > 1/\rho(G, H)$, then a.a.s. $G(n, p)$ does not have the (G, H)-extension property.

It is not difficult to see that Theorem 2 implies finiteness of spectra of all existence and extension sentences (see Section 4).

The next step is to consider sentences in prenex normal form that have 2 alternations. We call ϕ *a double-extension sentence*, if the sequence of quantifiers in ϕ is of the form $\forall \dots \forall \exists \dots \exists \forall \dots \forall$ (the respective properties are called *double-extension* as well). An asymptotical behavior of probabilities of the random graph double-extension properties was studied in [1,16].

Let W, G, H be three graphs such that $H \subset G \subset W$, $V(H) = \{a_1, \dots, a_s\}$, $V(G) \setminus V(H) = \{b_1, \dots, b_m\}$, $V(W) \setminus V(G) = \{c_1, \dots, c_r\}$, $s \ge 0$, $r, m \ge 1$. Assume that in W there are edges between each connected component of $W|_{\{c_1,\dots,c_r\}}$ and $W|_{\{b_1,\dots,b_m\}}$. Let $\mathcal{W}$ be a finite set of graphs such that all $W \in \mathcal{W}$ satisfy the above conditions (but r depends on W). We say that a graph has *the* $(\mathcal{W}, G, H)$*-double-extension property*, if, for any subset $\{y_1, \dots, y_s\}$ of distinct vertices in the graph, there exist distinct vertices $x_1, \dots, x_m$ such that, for all $W \in \mathcal{W}$ and all distinct vertices $z_1, \dots, z_{r(W)}$ the following happens:

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