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Independent Roman {2}-domination in graphs

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ABSTRACT

For a graph $G = (V, E)$, a Roman {2}-dominating function (R2DF) $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists an adjacent vertex, a neighbor $u \in N(v)$, with $f(u) = 2$, or at least two neighbors $x, y \in N(v)$ having $f(x) = f(y) = 1$. The weight of a R2DF is the sum $f(V) = \sum_{v\in V} f(v)$. A R2DF $f = (V_0, V_1, V_2)$ is called independent if $V_1 \cup V_2$ is an independent set. The independent Roman {2}-domination number $i_{\{R2\}}(G)$ is the minimum weight of an IR2DF on G. In this paper, we show that the decision problem associated with $i_{\{R2\}}(G)$ is NP-complete even when restricted to bipartite graphs. Then we show that for every graph G of order n, $0 \le i_{r2}(G) - i_{\{R2\}}(G) \le n/5$ and $0 \le i_R(G) - i_{\{R2\}}(G) \le n/4$, where $i_{r2}(G)$ and $i_R(G)$ are the independent 2-rainbow domination and independent Roman domination numbers, respectively. Moreover, we prove that the equality $i_{\{R2\}}(G) = i_{r2}(G)$ holds for trees.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The *open neighborhood* $N(v)$ of a vertex v consists of the vertices adjacent to v and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of v is the cardinality of its open neighborhood. By $\Delta = \Delta(G)$, we denote the *maximum degree* of the graph G. If S is a subset of $V(G)$ then $N(S) = \cup_{x\in S} N(x)$, $N[S] = \cup_{x\in S} N[x]$ and the subgraph induced by S in G is denoted $G[S]$. We say that a vertex $v \in S$ has a *private neighbor* with respect to S if $N[v] - N[S - \{v\}] \neq \emptyset$.

A *tree* is an acyclic connected graph. We write K_n for the *complete graph* of order n, P_n for the *path* of order n, and C_n for the *cycle* of order n.

Let f be a function that assigns to each vertex a subset of colors chosen from the set $\{1, 2\}$; that is $f : V(G) \to \mathcal{P}(\{1, 2\})$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have $\bigcup_{u\in N(v)} f(u) = \{1, 2\}$, then f is called a 2*-rainbow dominating function* (2**RDF**) of G. The weight of a 2RDF f is defined as $f(V) = \sum_{v\in V(G)} |f(v)|$. For a sake of simplicity, a 2RDF f on a graph G will be represented by the ordered partition $(V^f_\emptyset, V^f_{\{1\}}, V^f_{\{2\}}, V^f_{\{1,2\}})$ of $V(G)$ induced by f, where $V^f_\emptyset = \{u \in V \mid f(u) = \emptyset\}$, $V^f_{\{1\}} = \{u \in V \mid f(u) = \{1\}\}$, $V^f_{\{2\}} = \{u \in V \mid f(u) = \{2\}\}$ and $V^f_{\{1,2\}} = \{u \in V \mid f(u) = \{1, 2\}\}$. A function $f : V(G) \to \mathcal{P}(\{1, 2\})$ is called an *independent* 2*-rainbow dominating function* (**I** 2**RDF**) of G, if f is a 2RDF and no two vertices in $V(G) - V^f_\emptyset$ are adjacent. The *independent* 2*-rainbow domination number* $i_{r2}(G)$ is the minimum weight of an I2RDF of G. 2-Rainbow domination was introduced by Brešar et al. in [1], and has been studied by several authors, for example, see [2,3,5,6,13–15].

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* **(RDF)** on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V) = \sum_{u\in V(G)} f(u)$. A function

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$f: V(G) \to \{0, 1, 2\}$ in called *independent Roman dominating function* (**IRDF**) if f is an RDF and no two vertices assigned positive values are adjacent. The *independent Roman domination number* $i_R(G)$ is the minimum weight of an IRDF on G. Roman domination was introduced by Cockayne et al. [7] in 2004 and was inspired by the work of ReVelle and Rosing [11] and Stewart [12]. It is worth mentioning that since its introduction, around hundred papers have been published on various aspects of Roman domination in graphs, for example, see [3,4,9,10].

In a recent paper, the authors [4] introduced a new variant of Roman dominating functions. For a graph $G = (V, E)$, a *Roman* $\{2\}$*-dominating function* $f: V \to \{0, 1, 2\}$ has the slightly different property that only for every vertex $v \in V$ with $f(v) = 0, f(N(u)) \geq 2$, that is, either there exists a neighbor $u \in N(v)$, with $f(u) = 2$, or at least two neighbors $x, y \in N(u)$ have $f(x) = f(y) = 1$. The weight of a Roman $\{2\}$-dominating function is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a Roman $\{2\}$-dominating function f is the Roman $\{2\}$-domination number, denoted $\gamma_{\{R2\}}(G)$. Let $f = (V_0, V_1, V_2)$ be a function $f: V(G) \to \{0, 1, 2\}$ on a graph $G = (V, E)$, where $V_i = \{v | f(v) = i\}$, for $i \in \{0, 1, 2\}$.

In this paper, we are interested in Roman $\{2\}$-dominating functions $f = (V_0, V_1, V_2)$ for which $V_1 \cup V_2$ is an independent set. Such functions are called *independent Roman* $\{2\}$ *-dominating functions* (**IR2DF**). The first question that may arise, do all graphs have independent Roman $\{2\}$ -dominating functions? The following result provides a positive answer to this question.

Proposition 1. *Every graph G has an independent Roman* $\{2\}$*-dominating function.*

Proof. Let S be a maximal independent set of G. Let A be the set of vertices in S that have private neighbors in $V - S$ and let $B = S - A$. Observe that every vertex of $V - S$ is either a private neighbor of some vertex of A has at least two neighbors in $A \cup B$. Define a function f on G by assigning the value 2 to every vertex in A, the value 1 to vertex in B and the value 0 to the remaining vertices. Clearly, f is an independent Roman $\{2\}$-dominating function on G which proves the result. □

According to Proposition 1, we define the *independent Roman* $\{2\}$*-domination number*, denoted by $i_{\{R2\}}(G)$, as the minimum weight of an IR2DF on G.

In this paper, we initiate the study of independent Roman $\{2\}$-domination. We first show that the decision problem associated with $i_{\{R2\}}(G)$ is NP-complete for bipartite graphs. Then we prove that for every graph G of order n, $0 \leq i_{r2}(G) - i_{\{R2\}}(G) \leq n/5$ and $0 \leq i_R(G) - i_{\{R2\}}(G) \leq n/4$. Moreover, we prove that $i_{\{R2\}}(T) = i_{r2}(T)$ for every tree T. But before presenting these results, let us give the following observation.

Observation 2. *For every graph G,* $i_{\{R2\}}(G) \leq i_{r2}(G) \leq i_R(G)$.

Proof. The inequality $i_{r2}(G) \leq i_R(G)$ can be found in [6]. Let $f = (V^f_\emptyset, V^f_{\{1\}}, V^f_{\{2\}}, V^f_{\{1,2\}})$ be an $i_{r2}(G)$-function. Define the function g on G by $g(x) = 2$ if $x \in V^f_{\{1,2\}}$, $g(x) = 1$ if $x \in V^f_{\{1\}} \cup V^f_{\{2\}}$ and $g(x) = 0$ if $x \in V^f_\emptyset$. Clearly g is an IR2DF on G with weight $i_{r2}(G)$, and so $i_{\{R2\}}(G) \leq i_{r2}(G)$. □

The complete graph K_n is a simplest example for which the equality holds between the three parameters. Moreover, these parameters can also have distinct values. To see consider the connected graph G obtained from two disjoint cycles C_6 by adding one edge joining a vertex of one cycle to a vertex of the other cycle. Then $i_{\{R2\}}(C_6) = 6$, $i_{r2}(C_6) = 7$ and $i_R(C_6) = 8$.

2. Complexity result

In this section, we consider the decision problem associated with independent Roman $\{2\}$-dominating functions.

INDEPENDENT ROMAN $\{2\}$-DOMINATING FUNCTION (IR2D)
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Does G have an independent Roman $\{2\}$-dominating function of weight at most k?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, Exact-3-Cover (X3C), to IR2D.

EXACT 3-COVER (X3C)
Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
Question: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

Theorem 3. *IR2D is NP-complete for bipartite graphs.*

Proof. IR2D is a member of $\mathcal{NP}$, since we can check in polynomial time that a function $f: V \to \{0, 1, 2\}$ has weight at most k and is an independent Roman $\{2\}$-dominating function. Now let us show how to transform any instance of X3C into an instance G of IR2D so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we create a path $P_2: w_i$-z_i. Let $W = \{w_1, w_2, \ldots, w_{3q}\}$ and $Z = \{z_1, z_2, \ldots, z_{3q}\}$. For each $C_j \in C$ we build a graph H_j obtained from two cycles C_4 by adding a new vertex c_j attached to one vertex of each cycle C_4. Let us denote the vertices of the two cycles by u^j_{i1}-u^j_{i2} -u^j_{i3}-u^j_{i4}-u^j_{i1}, where $i \in \{1, 2\}$. Without loss of generality, we assume that c_j is adjacent to u^j_{11} and u^j_{21}. Let $Y = \{c_1, c_2, \ldots, c_t\}$. Now to obtain a graph G, we add edges $c_j w_i$ if $x_i \in C_j$. Clearly G is a bipartite graph.

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