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# Extremal hypergraphs for matching number and domination number[☆]

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ABSTRACT

A matching in a hypergraph $\mathcal{H}$ is a set of pairwise disjoint hyperedges. The matching number $\nu(\mathcal{H})$ of $\mathcal{H}$ is the size of a maximum matching in $\mathcal{H}$. A subset $D$ of vertices of $\mathcal{H}$ is a dominating set of $\mathcal{H}$ if for every $v \in V \setminus D$ there exists $u \in D$ such that $u$ and $v$ lie in an hyperedge of $\mathcal{H}$. The cardinality of a minimum dominating set of $\mathcal{H}$ is the domination number of $\mathcal{H}$, denoted by $\gamma(\mathcal{H})$. It was proved that $\gamma(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$ for $r$-uniform hypergraphs and the 2-uniform hypergraphs (graphs) achieving equality $\gamma(\mathcal{H}) = \nu(\mathcal{H})$ have been characterized. In this paper we generalize the inequality $\gamma(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$ to arbitrary hypergraph of rank $r$ and we completely characterize the extremal hypergraphs $\mathcal{H}$ of rank 3 achieving equality $\gamma(\mathcal{H}) = (r-1)\nu(\mathcal{H})$.



## 1. Introduction

Hypergraphs are a natural generalization of undirected graphs in which "edges" may consist of more than 2 vertices. More precisely, a *(finite) hypergraph* $\mathcal{H} = (V, E)$ consists of a (finite) set $V$ and a collection $E$ of non-empty subsets of $V$. The elements of $V$ are called *vertices* and the elements of $E$ are called *hyperedges*, or simply *edges* of the hypergraph. If there is a risk of confusion we will denote the vertex set and the edge set of a hypergraph $\mathcal{H}$ explicitly by $V(\mathcal{H})$ and $E(\mathcal{H})$, respectively. A hypergraph $\mathcal{H} = (V, E)$ is *simple* if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E$. Throughout this paper, we only consider simple hypergraphs.

The *rank* of a hypergraph $\mathcal{H}$ is $r(\mathcal{H}) = \max_{e\in E}|e|$. An *r-edge* in $\mathcal{H}$ is an edge of size $r$. The hypergraph $\mathcal{H}$ is said to be *r-uniform* if every edge of $\mathcal{H}$ is an $r$-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs.

A *matching* in a hypergraph $\mathcal{H}$ is a set of disjoint hyperedges. The *matching number*, denoted by $\nu(\mathcal{H})$, of a hypergraph $\mathcal{H}$ is the size of a maximum matching in $\mathcal{H}$.

A subset $D$ of $V(\mathcal{H})$ is called a *dominating set* of $\mathcal{H}$ if for every $v \in V(\mathcal{H}) \setminus D$ there exists $u \in D$ such that $u$ and $v$ lie in an hyperedge of $\mathcal{H}$. The minimum cardinality of a dominating set of $\mathcal{H}$ is called its *domination number*, denoted by $\gamma(\mathcal{H})$. Dominating sets are important objects in communication networks, as they represent parts from which the entire network can be reached directly. Messages can then be transmitted from sources to destinations via such a "backbone" with suitably chosen links. The literature on domination has been surveyed and detailed in [13,14,18]. Domination in hypergraphs was introduced by Acharya [1] and studied further in [2,6,15,20].

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A subset $T$ of vertices in a hypergraph $\mathcal{H}$ is a *transversal* (also called *cover* or *hitting set* in many papers) if $T$ has a nonempty intersection with every edge of $\mathcal{H}$. The *transversal number*, $\tau(\mathcal{H})$, of $\mathcal{H}$ is the minimum size of a transversal of $\mathcal{H}$. Transversals in hypergraphs are extensively studied in the literature (see, for example, [4,7,9,16,17,23]).

By definition, it is easy to see that any transversal of a hypergraph $\mathcal{H}$ without isolated vertex is a dominating set of $\mathcal{H}$ and it must meet all edges of a maximum matching of $\mathcal{H}$. Furthermore, note that the union of the edges of a maximal matching in $\mathcal{H}$ obviously forms a transversal. Hence the transversal number of $\mathcal{H}$ is at most $r$ times its matching number. We state these relationships among the transversal number, the domination number and the matching number in hypergraphs as an observation.

**Observation 1.1.** *For a hypergraph $\mathcal{H}$ of rank $r$ without isolated vertex, $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$, $\gamma(\mathcal{H}) \leq \tau(\mathcal{H})$, and $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$.*

Arumugam et al. [5] investigated the hypergraphs satisfying $\gamma(\mathcal{H}) = \tau(\mathcal{H})$, and proved that their recognition problem is NP-hard already on the class of linear hypergraphs of rank 3. A long-standing conjecture, known as Ryser's conjecture, asserts that $\tau(\mathcal{H}) \leq (k-1)\nu(\mathcal{H})$ for a $k$-partite hypergraph $\mathcal{H}$ (see, e.g. [3,12]). The conjecture turns to be notoriously difficult and remains open for $k \geq 4$. The relationship between the parameters $\tau(\mathcal{H})$ and $\nu(\mathcal{H})$ in hypergraphs has also been studied in [8,10,19].

In particular, if a hypergraph is 2-uniform, that is, it is a (simple) graph, then the following inequality chain is well-known.

**Theorem 1.1** (*[14]*)**.** *If $G$ is a graph without isolated vertex, then $\gamma(G) \leq \nu(G) \leq \tau(G)$.*

We observed in [21] that the above inequality chain does not hold in hypergraphs and the difference $\gamma(\mathcal{H}) - \nu(\mathcal{H})$ may be arbitrarily large for hypergraphs $\mathcal{H}$ with rank $r \geq 3$. However, we can extend the inequality $\gamma(G) \leq \nu(G)$ for graphs to uniform hypergraphs in [21] as follows: for an $r$-uniform hypergraph $\mathcal{H}$ with no isolated vertex, $\gamma(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$, and this bound is sharp. This paper further observes that the inequality still holds for arbitrary hypergraphs of rank $r$.

**Theorem 1.2.** *If $\mathcal{H}$ is a hypergraph of rank $r$ $(\geq 2)$ without isolated vertex, then $\gamma(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$.*

In general, a constructive characterization of extremal hypergraphs of rank $r$ achieving equality in Theorem 1.2 seems difficult to obtain. For 2-uniform hypergraphs, i.e., graphs, Kano et al. [22] gave a complete characterization for extremal graphs with the equality by giving a characterization of star-uniform graphs and showing that a graph $G$ is star-uniform if and only if $\gamma(G) = \nu(G)$. In Section 4, we will provide a complete characterization of extremal hypergraphs of rank 3 with equality $\gamma(\mathcal{H}) = (r-1)\nu(\mathcal{H})$.

## 2. Terminology and notation

Let us introduce more definitions and notations. Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph. Two vertices $u$ and $v$ are *adjacent* in $\mathcal{H}$ if there is an edge $e \in E(\mathcal{H})$ such that $u, v \in e$. A vertex $v$ and an edge $e$ of $\mathcal{H}$ are *incident* if $v \in e$. The *degree* of a vertex $v \in V(\mathcal{H})$, denoted by $d_{\mathcal{H}}(v)$ or simply by $d(v)$ if $\mathcal{H}$ is clear from the context, is the number of edges incident to $v$. A vertex of degree zero is called an *isolated vertex*. A vertex of degree $k$ is called a *degree-$k$ vertex*. Let $r, n$ be integers, $1 \leq r \leq n$. We define the *$r$-uniform complete hypergraph* on $n$ vertices (or the *$r$-complete hypergraph*) to be a hypergraph, denoted by $K_n^r$, consisting of all the $r$-subsets of a set of cardinality $n$.

A *partial hypergraph* $\mathcal{H}' = (V', E')$ of a hypergraph $\mathcal{H} = (V, E)$, denoted by $\mathcal{H}' \subseteq \mathcal{H}$, is a hypergraph such that $V' \subseteq V$ and $E' \subseteq E$. In the class of graphs, partial hypergraphs are called *subgraphs*. In particular, if $V' = V$, $\mathcal{H}'$ is called a *spanning partial hypergraph* of $\mathcal{H}$. The partial hypergraph $\mathcal{H}' = (V', E')$ is *induced* if $E' = \{e \in E \mid e \subseteq V'\}$. Induced hypergraphs will be denoted by $\mathcal{H}[V']$.

Two vertices $u$ and $v$ in $\mathcal{H}$ are *connected* if there is a sequence $u = v_0, v_1, \ldots, v_k = v$ of vertices of $\mathcal{H}$ in which $v_{i-1}$ is adjacent to $v_i$ for $i = 1, 2, \ldots, k$. A *connected hypergraph* is a hypergraph in which each pair of vertices is connected. A maximal connected partial hypergraph of $\mathcal{H}$ is a *connected component* of $\mathcal{H}$. Thus, no edge in $\mathcal{H}$ contains vertices from different components.

## 3. Proof of Theorem 1.2

A direct proof of Theorem 1.2 is short. For the sake of the characterization in next section, we need to include the proof.

**Proof of Theorem 1.2.** We may assume that $\mathcal{H}$ is connected (otherwise we treat each connected component separately). Let $\mathcal{H}^*$ be a hypergraph obtained from $\mathcal{H}$ by successively deleting edges of $\mathcal{H}$ that do not contain any vertices of degree one in the resulting hypergraph at each stage. It is clear that $r(\mathcal{H}^*) \leq r(\mathcal{H}) = r$. When $\mathcal{H}$ is transformed to $\mathcal{H}^*$, note that isolated vertices cannot arise and the domination number cannot decrease, and every edge of $\mathcal{H}^*$ contains at least one degree-1 vertex, so every dominating set of $\mathcal{H}^*$ is a transversal of $\mathcal{H}^*$. Hence $\tau(\mathcal{H}^*) = \gamma(\mathcal{H}^*) \geq \gamma(\mathcal{H})$.

Let $M = \{e_1, e_2, \ldots, e_l\}$ be a maximum matching of $\mathcal{H}^*$. Then $\nu(\mathcal{H}^*) = |M|$. Let $e_i'$ be the set of vertices obtained from $e_i$ by deleting the vertices of degree-1 in $\mathcal{H}^*$. We claim that $D = \cup_{i=1}^{l} e_i'$ is a dominating set of $\mathcal{H}^*$. Indeed, for any vertex $x \in V(\mathcal{H}^*) \setminus D$, $\mathcal{H}^*$ has a hyperedge $e$ containing $x$. By the maximality of $M$, $e$ must intersect $V(M)$. This implies that $e \cap D \neq \emptyset$. Thus $D$ is a dominating set of $\mathcal{H}^*$. Hence $\gamma(\mathcal{H}^*) \leq |D| \leq (r-1)|M| = (r-1)\nu(\mathcal{H}^*)$. On the other hand, note that every maximum matching of $\mathcal{H}^*$ is a matching of $\mathcal{H}$, so $\nu(\mathcal{H}^*) \leq \nu(\mathcal{H})$. Therefore, $\gamma(\mathcal{H}) \leq \gamma(\mathcal{H}^*) \leq (r-1)\nu(\mathcal{H}^*) \leq (r-1)\nu(\mathcal{H})$. □

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