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Some problems on induced subgraphs

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ABSTRACT

We discuss some problems related to induced subgraphs.

(1) A good upper bound for the chromatic number in terms of the clique number for graphs in which every induced cycle has length 3 or 4.
(2) The perfect chromatic number of a graph, which is the smallest number of perfect sets into which the vertex set of a graph can be partitioned. (A set of vertices is said to be perfect if it induces a perfect graph.)
(3) Graphs in which the difference between the chromatic number and the clique number is at most one for every induced subgraph of the graph.
(4) A weakening of the notorious Erdős–Hajnal conjecture.
(5) A conjecture of Gyárfás about the χ-boundedness of a particular class of graphs.

All graphs considered in this article are simple, finite, and undirected. Let G be a graph. A graph that can be obtained from G by deleting some of its vertices is called an induced subgraph of G. A *hole* in a graph is an induced cycle of length at least 4. There are several notions of containment for graphs, and induced subgraph is the strongest one. We say that G "contains" a graph H when H is an induced subgraph of G. A class of graphs is said to be *hereditary* if every induced subgraph of every graph in the class is also in the class. (Authors who are interested in both subgraphs and induced subgraphs sometimes use the term "induced hereditary class" for what is called "hereditary" class here.) The chromatic number of a graph G is denoted by $\chi(G)$, the clique number by $\omega(G)$, and the stability number by $\alpha(G)$. A graph G is called *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. The line graph of a graph G, denoted by $L(G)$, is the graph with vertex set $E(G)$ and two vertices are adjacent if they are adjacent edges in G. A hereditary class $\mathcal{G}$ of graphs is said to be χ-bounded with χ-bounding function $f(x)$ if for every graph $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$.

1. Graphs in which every induced cycle is a triangle or a square

A graph in which every induced cycle is a triangle is called a chordal graph. (There are several other names, like triangulated graph, rigid circuit graph, and perfect elimination graph.) A graph in which every induced cycle is a square is called a chordal bipartite graph. Both classes are perfect: Chordal graphs was the first interesting class proved to be perfect in the late 1950s, and chordal bipartite graphs are bipartite, and hence perfect. What if every induced cycle in a graph is either a triangle or a square? Such graphs need not be perfect. The complement of a 7-cycle is an example.

We prove that the class of graphs not containing holes of length at least 5 is χ-bounded by the function $f(x) = 2^{2^x}$. The proof uses the leveling argument and several ideas from the recent paper by Scott and Seymour [14]. The main point here is that since we are forbidding all holes of length at least 5, we can bypass their "spine", "parent rule", and "parity property". Also, it gives a slightly better bound, although not in the final χ-bounding function.

A leveling L is a sequence of disjoint subsets of vertices $(L_0, L_1, \ldots, L_k)$ such that $|L_0| = 1$ and every vertex in L_i has a neighbor in L_{i-1}, and no vertex in L_i has a neighbor in L_j for $j < i - 1$.

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Lemma 1. *Suppose that every graph with no holes of length at least 5 and clique number at most $\omega - 1$ has chromatic number at most n. Let G be a graph with no holes of length at least 5 and clique number ω. Then $\chi(G) \leq 4n^2$.*

Proof. We may assume that G is connected. Let $v \in V(G)$. Let L_i be the set of vertices at distance i from v. The proof is by analyzing this leveling, in particular, looking at a level and the previous two levels. We show that for every k, $\chi(L_k) \leq 2n^2$. This is trivially true for $k = 0$. Consider $k = 1$. In fact, L_1 is the set of neighbors of the vertex in L_0, and hence has clique number at most $\omega - 1$, and hence chromatic number at most n. Now let $k \geq 2$. We will show how to color the vertices in L_k with $2n^2$ colors. Since we can use the same set of colors for each component of L_k, we may assume that L_k is connected. By deleting vertices in $L_0, L_1, \dots, L_{k-1}$ that do not have a child for which it is the only parent, we may assume every vertex in $L_0, L_1, \dots, L_{k-1}$ has a child for which it is the only parent. (Note that we are coloring only vertices in L_k, and deleting vertices in the previous levels does not change the chromatic number of $G[L_k]$.) Let $x \in L_{k-2}$. Let y be a child of x such that x is its only parent. We partition $L_{k-1} - \{y\}$ into A, B where A is the set of vertices in L_{k-1} that are neighbors of y, and $B = L_{k-1} - A - \{y\}$. Note that A has clique number at most $\omega - 1$, and hence chromatic number at most n. Suppose there is a vertex $z \in B$ that is not adjacent to x. Let a be a parent of z. Now there is a path between a and x with interior in $L_0 \cup L_1 \cup \cdots \cup L_{k-3}$, let P be a shortest such path. Also, there is a path between y and z with interior in L_k (this is guaranteed by the connectedness of L_k), let P' be a shortest such path. Now $a - P - x - y - P' - z - a$ is a hole of length at least 5, which is impossible. Hence we conclude that x is adjacent to every vertex in B. Hence $B \cup \{y\}$ has clique number at most $\omega - 1$, and chromatic number at most n. By using different colors for A and $B \cup \{y\}$, we see that L_{k-1} can be colored with at most $2n$ colors. Now partition L_k into sets $A_1, \dots, A_{2n}$ as follows: A vertex is in A_i if i is the smallest of the colors of its neighbors in L_{k-1}.

We will show that each A_i has clique number at most $\omega - 1$. Let K be a clique in some A_i. Suppose there exist $u, v \in K$ such that each has a parent of color i that is not a parent of the other (say u' is a parent of u but not v, v' is a parent of v but not u. Note that uv is an edge (they belong to a clique) and $u'v'$ is a non-edge (both u' and v' received the same color in a proper coloring). Let P be a shortest path between u' and v' with interior vertices in $L_0 \cup \cdots \cup L_{k-2}$. Now $u - v - v' - P - u' - u$ is a hole of length at least 5. Hence for any two vertices $u, v \in K$ every parent of u of color i is also a parent of v (or vice versa). Thus the set of parents of color i of vertices in K form a chain, and hence there must be a vertex of color i adjacent to every vertex in K. Hence $|K| \leq \omega - 1$. Thus each A_i has clique number at most $\omega - 1$, and hence chromatic number at most n. By using different set of colors for different A_i, we conclude that $\chi(L_k) \leq 2n^2$. By using one set of colors for odd levels and another set for even levels, we conclude that $\chi(G) \leq 4n^2$. □

Theorem 2. *Let G be a graph with no holes of length at least 5. Then $\chi(G) \leq 2^{2^{\omega(G)}}$.*

Proof. We claim that if G is a graph with no holes of length at least 5, then $\chi(G) \leq (\frac{1}{4})2^{2^{\omega(G)}}$. The proof is by induction on $\omega(G)$. The base case $\omega = 1$ is trivial. Suppose the statement is true for some k i.e., every graph with no holes of length at least 5 and clique number k has chromatic number at most $(\frac{1}{4})2^{2^k}$. Let G be a graph with no holes of length at least 5 and clique number $k + 1$. By Lemma 1, $\chi(G) \leq 4((\frac{1}{4})2^{2^k})^2 = (\frac{1}{4})2^{2^{k+1}}$. This completes the induction step. □

Scott and Seymour [14] proved almost the same bound for a much bigger class, viz. graphs not containing odd holes. The proof here mimics their proof but is much easier because of the stronger hypothesis. Note that the graphs mentioned in [14] have no holes of size at least 5 and have $\chi > \omega^{\frac{\log 3.5}{\log 3}}$. Their example is as follows: Let G_0 have one vertex, and for $k \geq 1$ let G_k be obtained from G_{k-1} by substituting a seven-vertex antihole for each vertex. Then G_k has no hole of length at least 5, $\omega(G_k) = 3^k$, and $\chi(G_k) \geq (\frac{7}{2})^k$. We conjecture the following.

Conjecture 3. *Let G be a graph with no holes of length at least 5. Then V(G) can be partitioned into two sets, none of them containing a maximum clique of G.*

Hoang and McDiarmid [10] conjecture that the previous statement actually holds for all odd-hole-free graphs. The truth of the above conjecture will immediately imply $\chi(G) \leq 2^{\omega(G)}$ for a graph G with no holes of length at least 5. It is possible that the following stronger conclusion holds.

Conjecture 4. *Let G be a graph with no holes of length at least 5. Then $\chi(G) \leq \omega(G)^2$.*

Alex Scott and Paul Seymour (private communication) have recently proved that $\chi(G) \leq 2^{\omega(G)^2}$ holds for every graph G with no holes of length at least 5.

Here is a numerical problem.

Problem 5. Let $f(n)$ be the largest chromatic number of a graph with no holes of length at least 5 and clique number n. Clearly $f(1) = 1$ and $f(2) = 2$. It is known from [3] that $f(3) = 4$. Determine $f(4)$.

Chordal graphs (see [4]), have fantastic properties, like the following:

- Has a simplicial ordering
- Every minimal cutset is a clique
- Intersection of subtrees of a tree