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Some properties and applications of odd-colorable r-hypergraphs[☆]

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ABSTRACT

Let $r \geq 2$ and r be even. An r-hypergraph G on n vertices is called odd-colorable if there exists a map $\varphi : [n] \to [r]$ such that for any edge $\{j_1, j_2, \ldots, j_r\}$ of G, we have $\varphi(j_1) + \varphi(j_2) + \cdots\cdot + \varphi(j_r) \equiv r/2 (\bmod r)$. In this paper, we first determine that, if $r = 2^q(2t+1)$ and $n \geq 2^q(2^q-1)r$, then the maximum chromatic number in the class of the odd-colorable r-hypergraphs on n vertices is 2^q, which answers a question raised by V. Nikiforov recently in Nikiforov (2017). We also study some applications of the spectral symmetry of the odd-colorable r-hypergraphs given in the same paper by V. Nikiforov. We show that the Laplacian spectrum $Spec(\mathcal{L}(G))$ and the signless Laplacian spectrum $Spec(\mathcal{Q}(G))$ of an r-hypergraph G are equal if and only if r is even and G is odd-colorable. As an application of this result, we give an affirmative answer for the remaining unsolved case $r \not\equiv 0\ (mod\ 4)$ of a question raised in Shao et al. (2015) about whether $Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G))$ implies that $\mathcal{L}(G)$ and $\mathcal{Q}(G)$ have the same H-spectrum.

1. Introduction

Denote the set $\{1, 2, \cdots\cdot, n\}$ by $[n]$. An r-hypergraph $G = (V(G), E(G))$ on n vertices is an r-uniform hypergraph each of whose edges contains exactly r vertices [1]. In this paper, r-hypergraph is simply called r-graph for convenience. A 2-graph is just an ordinary graph.

The definition of odd-coloring for tensors (they are called r-matrices in [10]) was introduced in [10], we just focus on its version for r-graphs as follows.

Definition 1. Let $r \geq 2$ and r be even. An r-graph G with $V(G) = [n]$ is called odd-colorable if there exists a map $\varphi : [n] \to [r]$ such that for any edge $\{j_1, j_2, \ldots, j_r\}$ of G, we have

$$\varphi(j_1) + \cdots\cdot + \varphi(j_r) \equiv r/2 (\bmod r).$$

The function φ is called an odd-coloring of G.

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The following concept of odd-bipartite r-graphs was taken from [6], and this concept acts as a generalization of the ordinary bipartite graphs.

Definition 2 (*[6]*). An r-graph $G = (V, E)$ is called odd-bipartite, if r is even and there exists some proper subset V_1 of V such that each edge of G contains exactly odd number of vertices in V_1.

The odd-bipartite r-graphs were also called odd-transversal r-graphs in the literature (see [1,3], or [10]). The connection between odd-bipartiteness and spectra of r-graphs was studied in [6–9] and [14].

In [10], it was proved that an odd-bipartite graph is always odd-colorable (see Proposition 11 of [10]), and furthermore, in the case $r \equiv 2 (\mathrm{mod}\ 4)$, G is odd-colorable if and only if G is odd-bipartite (see Proposition 12 of [10]).

An r-graph G is called k-chromatic if its vertices can be partitioned into k sets so that each edge intersects at least two sets. The chromatic number $\chi(G)$ of G is the smallest k for which G is k-chromatic. The chromatic number of an odd-colorable r-graph was also considered in [10]. Clearly, each nontrivial odd-bipartite graph has chromatic number 2. A family of 3-chromatic odd-colorable $4k$-graphs on n vertices was constructed in [10]. Notice that odd-colorable r-graphs are defined only for even r. For further information about the chromatic number of an odd-colorable graph, the following question was raised in [10].

Question 3. *Let* $r \equiv 0 (\mathrm{mod}\ 4)$. *What is the maximum chromatic number of an odd-colorable r-graph on n vertices?*

In Section 2, we will determine that, if r is even, $r = 2^q(2t + 1)$ for some integers q, t and $n \geq 2^q(2^q - 1)r$, then the maximum chromatic number in the class of the odd-colorable r-graphs on n vertices is 2^q. This result provides an answer to Question 3.

Definition 4 (*[7,12]*). Let $G = (V(G), E(G))$ be an r-graph on n vertices. The adjacency tensor of G is defined as the order r dimension n tensor $\mathcal{A}(G)$ whose $(j_1 \cdots j_r)$-entry is:

$$(\mathcal{A}(G))_{j_1 j_2 \cdots j_r} = \begin{cases} \dfrac{1}{(r-1)!} & \text{if } \{j_1, j_2, \ldots, j_r\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}(G)$ be an order r dimension n diagonal tensor, with its diagonal entry $\mathcal{D}_{jj\cdots j}$ being the degree of vertex j, for all $j \in [n]$. Then $\mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$ is called the Laplacian tensor of r-graph G, and $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ is called the signless Laplacian tensor of G.

The following general product of tensors, which was defined in [13], is a generalization of the matrix case. Let $\mathcal{A}$ and $\mathcal{B}$ be dimension n and order $m \geq 2$ and $k \geq 1$ tensors, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of dimension n and order $(m-1)(k-1)+1$ with entries:

$$\mathcal{C}_{i\alpha_1 \cdots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m \in [n_2]} \mathcal{A}_{i i_2 \cdots i_m} \mathcal{B}_{i_2 \alpha_1} \cdots \mathcal{B}_{i_m \alpha_{m-1}}, \tag{1}$$

where $i \in [n]$, $\alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}$.

Let $\mathcal{T}$ be an order r dimension n tensor and $x = (x_1, \cdots\cdot, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension n. Then by (1) $\mathcal{T}x$ is a vector in $\mathbb{C}^n$ whose jth component is

$$(\mathcal{T}x)_j = \sum_{j_2, \ldots, j_r = 1}^{n} \mathcal{T}_{j j_2 \cdots j_r} x_{j_2} \cdots x_{j_r}. \tag{2}$$

Let $x^{[r]} = (x_1^r, \ldots, x_n^r)^T$. Then (see [2] [12]) a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $\mathcal{T}$ of order r if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequations

$$\mathcal{T}x = \lambda x^{[r-1]}, \tag{3}$$

and in this case, x is called an eigenvector of $\mathcal{T}$ corresponding to the eigenvalue λ. The spectral radius of $\mathcal{T}$ is defined as

$$\rho(\mathcal{T}) = \max\{|\mu| : \mu \text{ is an eigenvalue of } \mathcal{T}\}.$$

In order to define the spectra of tensors, we first need to introduce the concept of the determinants of tensors. Originally the determinants of tensors were defined as the resultants of some corresponding system of homogeneous equations on n variables. Here we give the following equivalent definition for the determinants of tensors.

Definition 5. Let $\mathcal{A}$ be an order m dimension n tensor with $m \geq 2$. Then its determinant $det(\mathcal{A})$ is defined to be the unique polynomial on the entries of $\mathcal{A}$ satisfying the following three conditions:
(1) $det(\mathcal{A}) = 0$ if and only if the system of homogeneous equations $\mathcal{A}x = 0$ has a nonzero solution.
(2) $det(\mathcal{A}) = 1$, when $\mathcal{A} = \mathbb{I}$ is the unit tensor.
(3) $det(\mathcal{A})$ is an irreducible polynomial on the entries of $\mathcal{A}$, when the entries of $\mathcal{A}$ are viewed as distinct independent variables.