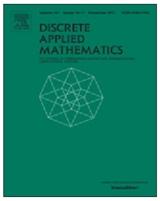
Discrete Applied Mathematics 236 (2018) 472–476

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam



Note

A note on a directed version of the 1-2-3 Conjecture

Mirko Horňák [a], Jakub Przybyło [b,*], Mariusz Woźniak [b]

[a] Institute of Mathematics, P.J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia
[b] AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland

ARTICLE INFO

Article history:
Received 16 January 2017
Received in revised form 2 November 2017
Accepted 15 November 2017
Available online 6 December 2017

Keywords:
Edge coloring
Digraph
1-2-3 Conjecture

ABSTRACT

The least k such that a given digraph $D = (V, A)$ can be arc-labeled with integers in the interval $[1, k]$ so that the sum of values in-coming to x is distinct from the sum of values out-going from y for every arc $(x, y) \in A$, is denoted by $\bar{\chi}_{\text{Ł}}^{e}(D)$. This corresponds to one of possible directed versions of the well-known 1-2-3 Conjecture. Unlike in the case of other possibilities, we show that $\bar{\chi}_{\text{Ł}}^{e}(D)$ is unbounded in the family of digraphs for which this parameter is well defined. However, if the family is restricted by excluding the digraphs with so-called lonely arcs, we prove that $\bar{\chi}_{\text{Ł}}^{e}(D) \leq 4$, and we conjecture that $\bar{\chi}_{\text{Ł}}^{e}(D) \leq 3$ should hold.

1. Introduction

The origins of the problem go back to the eighties of the twentieth century and are associated with attempts to define the notion of irregularity of a graph using labels (colors) on the edges of a graph. Among those attempts, it was the irregularity strength that attracted the greatest attention. Perhaps this was due to a simple "geometric" interpretation based on the fact that although each graph of order greater than one contains at least two vertices of the same degree, an analogous statement is not true for multigraphs, *i.e.*, graphs in which we allow more than one edge between two (distinct) vertices.

Let $G = (V, E)$ be a graph. Given an integer k, a k-edge-coloring (labeling) of G is a function $f : E \to \{1, 2, \dots, k\}$. For $x \in V$, we put $\sigma(x) = \sum_{e \ni x} f(e)$. We say that two vertices x, y are *sum-distinguished* (the coloring f is *sum-distinguishing*) if $\sigma(x) \neq \sigma(y)$. The *irregularity strength* of G is the minimum k such that there exists a k-edge-coloring f sum-distinguishing all vertices in the graph G. The coloring f can be represented by substituting each edge e by a multiedge with multiplicity $f(e)$. The sum $\sigma(x)$ of labels around a vertex x is then equal to the degree of x in the respective multigraph.

A k-edge-coloring f of G is called *neighbor-sum-distinguishing* if $\sigma(x) \neq \sigma(y)$ whenever xy is an edge of G (we refer to it as to an *nsd-coloring* for short). Such a local variant of the irregularity strength gained a great popularity in the twenty first century due to the following beautiful conjecture of Karoński, Łuczak, and Thomason [5], commonly called the **1-2-3 Conjecture** nowadays.

Conjecture 1. *If $G = (V, E)$ is a graph without isolated edges, then there is an nsd-coloring $f : E \to \{1, 2, 3\}$ of G.*

Following the notation from the survey paper by Seamone [7] we will denote the least k so that there is an nsd-k-edge-coloring of a graph G by $\chi_{\Sigma}^{e}(G)$. The 1-2-3 Conjecture thus presumes that $\chi_{\Sigma}^{e}(G) \leq 3$ for every graph G without isolated edges. The best currently known general upper bound stating that $\chi_{\Sigma}^{e}(G) \leq 5$ is due to Kalkowski, Karoński and Pfender [4]. The conjecture is verified for particular graph classes, *e.g.*, bipartite graphs, see [5].

* Corresponding author.
E-mail addresses: mirko.hornak@upjs.sk (M. Horňák), jakubprz@agh.edu.pl (J. Przybyło), mwozniak@agh.edu.pl (M. Woźniak).

https://doi.org/10.1016/j.dam.2017.11.016
0166-218X/

Theorem 2. *If G is a bipartite graph without isolated edges, then $\chi^e_\Sigma(G) \le 3$.* ■

We will focus on nsd-colorings of digraphs $D = (V, A)$, where we will use a simplified notation xy for an arc (x, y). Given a k-arc-coloring $f : A \to \{1, 2, \dots, k\}$ and a vertex $x \in V$, we discern *out-going arcs* $xy \in A$ and *in-coming arcs* $yx \in A$, and analogously the *out-sum* $\sigma^+(x) = \sum_{xy\in A} f(xy)$ and the *in-sum* $\sigma^-(x) = \sum_{yx\in A} f(yx)$ of x. Several variants of nsd-colorings of digraphs have already been considered.

The first problem of this type was introduced by Borowiecki, Grytczuk, and Pilśniak, and concerned so-called relative sums, defined for a vertex x as $\sigma_\pm(x) = \sigma^+(x) - \sigma^-(x)$. The least k so that a k-arc-coloring of a given digraph $D = (V, A)$ exists with $\sigma_\pm(x) \ne \sigma_\pm(y)$ for every arc $xy \in A$ is denoted by $\chi^e_\pm(D)$. The authors proved in [3] the sharp upper bound $\chi^e_\pm(D) \le 2$ valid for every digraph D.

Only just then Baudon, Bensmail, and Sopena considered the least integer k admitting a k-arc-coloring of a digraph $D = (V, A)$ such that $\sigma^+(x) \ne \sigma^+(y)$ for every $xy \in A$. We denote such k by $\chi^e_+(D)$. In [2] the authors showed that $\chi^e_+(D) \le 3$ for every digraph D and proved that given a digraph D, the problem of determining whether $\chi^e_+(D) \le 2$ is NP-complete. (Note that obviously we obtain the same thesis for the twin graph invariant $\chi^e_-(D)$ of the above one, where we require: $\sigma^-(x) \ne \sigma^-(y)$ for every $xy \in A$.)

The third natural variant was suggested by Łuczak [6], who proposed to study the sum-distinguishing requirement $\sigma^+(x) \ne \sigma^-(y)$ for $xy \in A$. Barme et al. [1] observed that the corresponding parameter $\chi^e_Ł(D)$ is not defined provided that D has an arc xy satisfying $d^+(x) = 1 = d^-(y)$, called a *lonely arc*. Nevertheless, they were able to prove the following upper bound.

Theorem 3. *If D is a digraph without lonely arcs, then $\chi^e_Ł(D) \le 3$.* ■

The proof of Theorem 3 is based on the equivalence between the inequality $\chi^e_Ł(D) \le k$ and the existence of an nsd-k-edge-coloring of a special (undirected) bipartite graph associated with D. Thus by the classification from the paper of Thomassen, Wu and Zhang [8], one may moreover determine $\chi^e_Ł(D)$ for any digraph D (without lonely arcs) in a polynomial time now.

In this note we study the inverse (in a way) of the problem of Łuczak above, requiring that $\sigma^-(x) \ne \sigma^+(y)$ for $xy \in A$ (which seems to be the last natural open issue in this new field). In the next section we discuss when the corresponding graph invariant $\bar{\chi}^e_Ł(D)$ is well defined, and, surprisingly, we prove that for those digraphs $\bar{\chi}^e_Ł(D)$ may be arbitrarily large. On the other hand, in Section 3 we show that $\bar{\chi}^e_Ł(D) \le 4$ if lonely arcs are additionally forbidden. Finally, in the last section we pose a conjecture that then $\bar{\chi}^e_Ł(D) \le 3$ should hold, and present a few rich families of digraphs supporting this new 1-2-3-Conjecture for digraphs.

2. Boundlessness of the inverse Łuczak's problem

We call a digraph $D = (V, A)$ *tractable* if for a suitable k there is a k-arc-coloring f of D such that for any arc $xy \in A$, $\sigma^-(x) \ne \sigma^+(y)$. The least such k for a tractable digraph D is denoted by $\bar{\chi}^e_Ł(D)$.

There are two obvious obstacles for tractability. Consider a k-arc-coloring f of a digraph $D = (V, A)$. For a vertex $x \in V$, we denote by $A^-(x)$ ($A^+(x)$) the set of arcs in D in-coming to x (out-going from x, respectively). An arc $xy \in A$ is called a *source–sink arc*, an *s-s arc* for short, if x is a source and y is a sink of D (*i.e.*, $d^-(x) = 0$ and $d^+(y) = 0$). Then, inevitably, $\sigma^-(x) = 0 = \sigma^+(y)$. The situation is similar if both arcs xy and yx belong to A and xy is an s-s arc in the digraph $D' = D - yx$. We then say that $\{xy, yx\}$ is a *source–sink edge* (an *s-s edge* for short). Then $A^-(x) = A^+(y) = \{yx\}$, and hence $\sigma^-(x) = f(yx) = \sigma^+(y)$. It is straightforward to see that if we forbid these two configurations in D (which requires $|V| \ge 3$), then $A^-(x) \ne A^+(y)$ for every arc $xy \in A$, and thus there exists a k-arc-coloring of D with $\sigma^-(x) \ne \sigma^+(y)$ for every $xy \in A$ for a sufficiently large k.

Proposition 4. *A digraph D is tractable if and only if D has neither s-s arcs nor s-s edges.* ■

The three parameters χ^e_+, χ^e_- and $\chi^e_Ł$ fulfill a correspondingly formulated 1-2-3-Conjecture. Is it the case for the parameter $\bar{\chi}^e_Ł$, too? The digraph D_4 drawn in Fig. 1, gives us a negative answer to this question.

First, observe that D_4 has neither an s-s arc nor an s-s edge. Consider an arc-coloring f of D_4 such that $\sigma^-(x) \ne \sigma^+(y)$ whenever xy is an arc of D_4. Let $f(x_1x_2) = a, f(x_3x_4) = b, f(x_5x_6) = c, f(x_7x_8) = d$. The digraph D_4 satisfies $A^+(x_{2i-1}) = \{x_{2i-1}x_{2i}\} = A^-(x_{2i})$, $i = 1, 2, 3, 4$. Moreover, for any i, j with $1 \le i < j \le 4$, the arc $x_{2i}x_{2j-1}$ belongs to D_4, and hence

$$f(x_{2i-1}x_{2i}) = \sigma^-(x_{2i}) \ne \sigma^+(x_{2j-1}) = f(x_{2j-1}x_{2j}).$$

Therefore, the colors a, b, c, d of the dashed arcs $x_{2i-1}x_{2i}$, $i = 1, 2, 3, 4$, are pairwise distinct, and so $\bar{\chi}^e_Ł(D_4) \ge 4$.

Proposition 5. *For any integer $k \ge 2$ there is a digraph D_k with $\bar{\chi}^e_Ł(D_k) \ge k$.*

Proof. Consider a digraph D_k with the vertex set $\{x_1, x_2, \dots, x_{2k}\}$ and the arc set $\bigcup_{i=1}^{k}\left(\{x_{2i-1}x_{2i}\} \cup \bigcup_{j=i+1}^{k}\{x_{2i}x_{2j-1}\}\right)$. Suppose that an l-arc-coloring $f : E(D_k) \to \{1, 2, \dots, l\}$ satisfies $\sigma^-(x_i) \ne \sigma^+(x_j)$ whenever $x_ix_j \in E(D_k)$. It is easy to see proceeding as above that then necessarily $l \ge k$. ■

Corollary 6. *The parameter $\bar{\chi}^e_Ł$ is not bounded from above by an absolute constant.* ■