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Bounds on the burning number

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ABSTRACT

Motivated by a graph theoretic process intended to measure the speed of the spread of contagion in a graph, Bonato et al. (Burning a Graph as a Model of Social Contagion, Lecture Notes in Computer Science 8882 (2014) 13-22) define the burning number $b(G)$ of a graph G as the smallest integer k for which there are vertices $x_1, \ldots, x_k$ such that for every vertex u of G, there is some $i \in \{1, \ldots, k\}$ with $\text{dist}_G(u, x_i) \le k-i$, and $\text{dist}_G(x_i, x_j) \ge j-i$ for every $i, j \in \{1, \ldots, k\}$.

For a connected graph G of order n, they prove that $b(G) \le 2\lceil\sqrt{n}\rceil - 1$, and conjecture $b(G) \le \lceil\sqrt{n}\rceil$. We show that $b(G) \le \sqrt{\frac{32}{19}\cdot\frac{n}{1-\epsilon}} + \sqrt{\frac{27}{19\epsilon}}$ and $b(G) \le \sqrt{\frac{12n}{7}} + 3 \approx 1.309\sqrt{n} + 3$ for every connected graph G of order n and every $0 < \epsilon < 1$. For a tree T of order n with n_2 vertices of degree 2, and $n_{\ge 3}$ vertices of degree at least 3, we show $b(T) \le \left\lceil\sqrt{(n+n_2)+\frac{1}{4}}+\frac{1}{2}\right\rceil$ and $b(T) \le \lceil\sqrt{n}\rceil + n_{\ge 3}$. Furthermore, we characterize the binary trees of depth r that have burning number $r+1$.

1. Introduction

Motivated by a graph theoretic process intended to measure the speed of the spread of contagion in a graph, Bonato, Janssen, and Roshanbin [3,4] define a *burning sequence* of a graph G as a sequence $(x_1, \ldots, x_k)$ of vertices of G such that

$$\forall u \in V(G) : \exists i \in [k] : \text{dist}_G(u, x_i) \le k - i \text{ and} \tag{1}$$

$$\forall i, j \in [k] : \text{dist}_G(x_i, x_j) \ge j - i, \tag{2}$$

where $[k]$ denotes the set of the positive integers at most k. Furthermore, they define the *burning number* $b(G)$ of G as the length of a shortest burning sequence of G.

A burning sequence is supposed to model the expansion of a fire within a graph. At each discrete time step, first a new fire starts at a vertex that is not already burning, and then the fire spreads from burning vertices to all their neighbors that are not already burning. Condition (1) ensures that putting fire to the vertices of a burning sequence $(x_1, \ldots, x_k)$ in the order $x_1, \ldots, x_k$, all vertices of G are burning after k steps. Condition (2) ensures that one never puts fire to a vertex that is already burning.

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We consider only finite, simple, and undirected graphs, and use standard terminology and notation [6]. For a graph G, a vertex u of G, and an integer k, let $N^k_G[u] = \{v \in V(G) : \mathrm{dist}_G(u, v) \leq k\}$. Note that $N^0_G[u] = \{u\}$ and $N^1_G[u] = N_G[u] = \{u\} \cup N_G(u)$.

With this notation (1) is equivalent to

$$V(G) = N^{k-1}_G[x_1] \cup N^{k-2}_G[x_2] \cup \cdots \cup N^0_G[x_k]. \tag{3}$$

As previously said, condition (2) is motivated by the considered graph process, which in each step puts fire to a vertex that is not already burning. Our first result is that condition (2) is redundant.

Lemma 1. *The burning number of a graph G is the minimum length of a sequence $(x_1, \ldots, x_k)$ of vertices of G satisfying (3).*

Proof. Let k be the minimum length of a sequence satisfying (3). By definition, $b(G) \geq k$. It remains to show equality. For a contradiction, suppose $b(G) > k$. Let the sequence $s = (x_1, \ldots, x_k)$ be chosen such that (3) holds, and $j(s) = \min\{j \in [k] : \mathrm{dist}_G(x_i, x_j) < j - i \text{ for some } i \in [j-1]\}$ is as large as possible. Since $b(G) > k$, the index $j(s)$ is well defined. Let $i(s) \in [j(s)-1]$ be such that $\mathrm{dist}_G(x_{i(s)}, x_{j(s)}) < j(s) - i(s)$. Since $k > j(s) - 1$, there is a vertex y in

$$V(G) \setminus \left(N^{(j(s)-1)-1}_G[x_1] \cup N^{(j(s)-1)-2}_G[x_2] \cup \cdots \cup N^0_G[x_{j(s)-1}]\right).$$

Since $N^{k-j(s)}_G[x_{j(s)}] \subseteq N^{k-i(s)}_G[x_{i(s)}]$, the sequence $s' = (x_1, \ldots, x_{j(s)-1}, y, x_{j(s)+1}, \ldots, x_k)$ satisfies (3) and $j(s') > j(s)$, which is a contradiction. □

In view of Lemma 1, the burning number can be considered a variation (but distinct from) of well known distance domination parameters [7]. For a graph G and an integer k, a set D of vertices of G is a *distance-k-dominating set* of G if $\bigcup_{x \in D} N^k_G[x] = V(G)$. The *distance-$k$-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a distance-k-dominating set of G.

The following bound on the distance-k-domination number will be of interest.

Theorem 2 (Meir and Moon [8])**.** *If G is a connected graph of order n at least $k+1$, then $\gamma_k(G) \leq \frac{n}{k+1}$.*

As observed in [3,4] the burning number can be bounded above in terms of the distance-k-domination number. In fact, if $\{x_1, \ldots, x_\gamma\}$ is a distance-k-dominating set of G, then

$$\begin{aligned} V(G) &= N^k_G[x_1] \cup N^k_G[x_2] \cup \cdots \cup N^k_G[x_\gamma] \\ &= N^{k+\gamma-1}_G[x_1] \cup N^{k+\gamma-2}_G[x_2] \cup \cdots \cup N^k_G[x_\gamma]. \end{aligned}$$

Appending any k vertices to the sequence $(x_1, \ldots, x_\gamma)$ yields a sequence of length $k + \gamma$ satisfying (3), which, by Lemma 1, implies $b(G) \leq \gamma_k(G) + k$. Using Theorem 2 and choosing $k = \lceil \sqrt{n} \rceil - 1$, this implies the following.

Theorem 3 (Bonato, Janssen, and Roshanbin [3,4])**.** *If G is a connected graph of order n, then $b(G) \leq 2\lceil \sqrt{n} \rceil - 1$.*

One of the most interesting open problems concerning the burning number is the following.

Conjecture 4 (Bonato, Janssen, and Roshanbin [3,4])**.** *If G is a connected graph of order n, then $b(G) \leq \lceil \sqrt{n} \rceil$.*

Since the path P_n of order n has burning number $\lceil \sqrt{n} \rceil$ [3,4], the bound in Conjecture 4 would be tight.

Let $\mathrm{rad}(G)$ denote the radius of a graph G. Since $V(G) = N^{\mathrm{rad}(G)}_G[x]$ for every connected graph G and every vertex x of G of minimum eccentricity, Lemma 1 implies the following.

Theorem 5 (Bonato, Janssen, and Roshanbin [4])**.** *If G is a connected graph, then $b(G) \leq \mathrm{rad}(G) + 1$.*

In the present note, we improve the bound of Theorem 3 by showing several upper bounds on the burning number, thereby contributing to Conjecture 4. Furthermore, we characterize the extremal binary trees for Theorem 5.

2. Results

We begin with two straightforward results that lead to a first improvement of Theorem 3, and rely on arguments that are typically used to prove Theorem 2. For a vertex u of a rooted tree T, let T_u denote the subtree of T rooted in u that contains u as well as all descendants of u. Recall that the height of T_u is the eccentricity of u in T_u.

The order of a graph G is denoted by $n(G)$.

Lemma 6. *Let T be a tree. If the non-negative integer d is such that $N^d_T[u] \neq V(T)$ for every vertex u of T, then there is a vertex x of T and a subtree T' of T with $n(T') \leq n(T) - (d+1)$ and $V(T) \setminus V(T') \subseteq N^d_T[x]$.*

Proof. Root T at a vertex r. Since $N^d_T[r] \neq V(T)$, the height of T is at least $d+1$. The desired properties follow for a vertex x such that T_x has height exactly d and the tree $T' = T - V(T_x)$. □