ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Domination parameters with number 2: Interrelations and algorithmic consequences

Flavia Bonomo [a,b,*], Boštjan Brešar [c,d], Luciano N. Grippo [e], Martin Milanič [f,g], Martín D. Safe [h,e]

[a] Universidad de Buenos Aires. Facultad de Ciencias Exactas y Naturales. Departamento de Computación, Buenos Aires, Argentina
[b] CONICET-Universidad de Buenos Aires. Instituto de Investigación en Ciencias de la Computación (ICC), Buenos Aires, Argentina
[c] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
[d] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[e] Instituto de Ciencias, Universidad Nacional de General Sarmiento, Los Polvorines, Buenos Aires, Argentina
[f] University of Primorska, UP IAM, Muzejski trg 2, SI-6000 Koper, Slovenia
[g] University of Primorska, UP FAMNIT, Glagoljaška 8, SI-6000 Koper, Slovenia
[h] Departamento de Matemática, Universidad Nacional del Sur, Bahía Blanca, Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 6 July 2016
Accepted 14 August 2017
Available online xxxx

Keywords:
Graph domination
Total domination
Rainbow domination
2-domination
Integer domination
Double domination
Split graph
Approximation algorithm
Inapproximability

ABSTRACT

In this paper, we study the most basic domination invariants in graphs, in which number 2 is intrinsic part of their definitions. We classify them upon three criteria, two of which give the following previously studied invariants: the weak 2-domination number, $\gamma_{w2}(G)$, the 2-domination number, $\gamma_2(G)$, the $\{2\}$-domination number, $\gamma_{\{2\}}(G)$, the double domination number, $\gamma_{\times 2}(G)$, the total $\{2\}$-domination number, $\gamma_{t\{2\}}(G)$, and the total double domination number, $\gamma_{t\times 2}(G)$, where G is a graph in which the corresponding invariant is well defined. The third criterion yields rainbow versions of the mentioned six parameters, one of which has already been well studied, and three other give new interesting parameters. Together with a special, extensively studied Roman domination, $\gamma_R(G)$, and two classical parameters, the domination number, $\gamma(G)$, and the total domination number, $\gamma_t(G)$, we consider 13 domination invariants in graphs. In the main result of the paper we present sharp upper and lower bounds of each of the invariants in terms of every other invariant, a large majority of which are new results proven in this paper. As a consequence of the main theorem we obtain new complexity results regarding the existence of approximation algorithms for the studied invariants, matched with tight or almost tight inapproximability bounds, which hold even in the class of split graphs.

1. Introduction

1.1. Prologue

A continuously growing interest in the area of graph domination, which arises from both practical applications and combinatorial challenges, has made the theory rather incoherent; two monographs surveying domination theory were published almost twenty years ago [46,47]. Due to a large number of domination-type concepts, it is not always easy to notice

* Corresponding author.
E-mail addresses: fbonomo@dc.uba.ar (F. Bonomo), bostjan.bresar@um.si (B. Brešar), lgrippo@ungs.edu.ar (L.N. Grippo), martin.milanic@upr.si (M. Milanič), msafe@uns.edu.ar (M.D. Safe).

http://dx.doi.org/10.1016/j.dam.2017.08.017

and appreciate some deep results that capture a broad aspect of the theory. Several results in domination theory have been in some sense rediscovered, because an approach that works for one concept can often be used with some slight adjustment for several other related concepts. We wish to make a step in the direction of making the situation more transparent, by classifying some of the most basic domination invariants, in which number 2 is involved in the definition. We make a comparison of their values in graphs between each pair of them, and as a consequence, since the discovered translations between parameters can be efficiently constructed, a general approach that joins some algorithmic and complexity issues on all of these concepts is established. In many cases our results imply that an algorithm for one invariant gives a good approximation algorithm for some other invariant; in addition, strong inapproximability results are inferred for almost all considered parameters, which hold even in the class of split graphs. (Let us mention that in [7] some connections between a (different and smaller) group of domination parameters has been established, yet the main focus was on claw-free graphs.)

1.2. Classification of parameters

The central focus of the paper is on several domination invariants of graphs, which have number 2 appearing in their definition (in particular, vertices must be dominated twice or using the sum of weights 2), and we can classify them upon three different criteria. The first criterion is the set of weights that are allowed to be assigned to vertices, which can be either $\{0, 1, 2\}$ or only $\{0, 1\}$ (in rainbow versions, which we will consider in parallel, these weights can be either $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ or only $\{\emptyset, \{a\}, \{b\}\}$). The second criterion distinguishes three possibilities with respect to the set of vertices that need to be dominated, and at the same time the type of neighborhoods, which are considered in domination. The possibilities are as follows: only vertices with weight 0 need to be dominated ('outer domination'), all vertices need to be dominated and vertices with a positive weight dominate their closed neighborhoods ('closed domination'), and finally all vertices need to be dominated and only open neighborhoods are dominated by vertices with positive weight ('open domination'). The following table shows the six concepts that arise from these two criteria, all of which have already been studied in the literature (in parenthesis a standard symbol of the corresponding graph invariant is written[1]):

	$\{0, 1, 2\}$	$\{0, 1\}$
Outer	Weak 2-domination (γ_{w2})	2-domination (γ_2)
Closed	$\{2\}$-domination ($\gamma_{\{2\}}$)	Double domination ($\gamma_{\times 2}$)
Open	Total $\{2\}$-domination ($\gamma_{t\{2\}}$)	Total double domination ($\gamma_{t\times 2}$)

The third criterion is based on the so-called rainbow variations of these parameters, and thus distinguishes domination parameters as being rainbow or not. This criterion is motivated by the concept known as k-rainbow domination introduced in [9]; in the case $k = 2$ the corresponding graph invariant was denoted by γ_{r2}, see, e.g., [10]. Note that in this paper the concept will be called rainbow weak 2-domination, and the invariant will be denoted by $\widetilde{\gamma}_{w2}$, suggesting that it is the rainbow counterpart of the concept of weak 2-domination, whose graph invariant is denoted by γ_{w2}. The k-rainbow domination (and 2-rainbow domination, in particular) has been considered in several papers [12–14,68,70,74,76], and is interesting also because of its strong connection with the domination of Cartesian products of graphs; in fact, some initial results on the 2-rainbow domination number in [45] were expressed in the terminology of domination of prisms. In this paper we are mainly concerned with its conceptual features, which initiates several other rainbow domination parameters. Intuitively speaking they are obtained as follows: weight 0 is replaced by the label $\emptyset$, weight 1 by labels $\{a\}$ and $\{b\}$, and weight 2 by the label $\{a, b\}$, while the conditions imposed by each parameter are meaningfully adjusted to the rainbow version. The main difference is that instead of the sum of values of weights, in a rainbow version one considers the union of labels, and also the condition of having weight 2 in a neighborhood corresponds to having label $\{a, b\}$.

Given a graph G its weak 2-domination number is denoted by $\gamma_{w2}(G)$, its 2-domination number by $\gamma_2(G)$, its $\{2\}$-domination number by $\gamma_{\{2\}}(G)$, its double domination number by $\gamma_{\times 2}(G)$, its total $\{2\}$-domination number by $\gamma_{t\{2\}}(G)$ and its total double domination number by $\gamma_{t\times 2}(G)$. (We remark that the notion of *weak 2-domination* appeared in the literature also under the name "weak 2-rainbow domination" [10].) By the above reasoning each of these parameters has its rainbow counter-part, which we will denote in a systematic way, by putting the symbol $\widetilde{}$ above γ, indicating that we are considering the rainbow version of the known concept. Two of the parameters among $\widetilde{\gamma}_2(G)$, $\widetilde{\gamma}_{w2}(G)$, $\widetilde{\gamma}_{\{2\}}(G)$, $\widetilde{\gamma}_{\times 2}(G)$, $\widetilde{\gamma}_{t\{2\}}(G)$ and $\widetilde{\gamma}_{t\times 2}(G)$ (namely $\widetilde{\gamma}_{\{2\}}(G)$ and $\widetilde{\gamma}_{t\{2\}}(G)$) turn out to be easily expressible by the known graph invariants, and we have thus not studied them any further. We believe that other four rainbow domination parameters are worth of consideration.

There is yet another well studied domination parameter, which involves number 2, but does not directly fit into the above frame. Nevertheless, the so-called *Roman domination*, introduced in [75] (see also [20,64]) has been considered in a number of papers, and is conceptually relevant also to our study. In the condition of the Roman dominating function, only the vertices with weight 0 must have in the neighborhood a vertex with weight 2, while there is no such restriction for the vertices with weight 1 and 2. Beside Roman domination, whose parameter in denoted by γ_R, we decided to include in our study also the two classical domination concepts, i.e., the *domination* and the *total domination*, denoted by γ and γ_t, respectively. Hence in our main result, see Table 2 (on p. 14), thirteen domination parameters are mutually compared. To stay within a reasonable length of the paper (and to stay in line with the basic classification presented in this paper) we do not consider other variations that also involve number 2 in their definitions. In particular, we do not consider the concepts that arise from basic parameters by imposing additional restrictions (such as paired domination [48], independent Roman domination [2], exact double domination [16], etc.).

[1] The total double domination was also denoted by $\gamma_{\times 2,t}$ in the literature, and was also called the double total domination.

Please cite this article in press as: F. Bonomo, et al., Domination parameters with number 2: Interrelations and algorithmic consequences, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.08.017.