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A note on the polytope of bipartite TSP

Gergely Kovács [a], Zsolt Tuza [b,c], Béla Vizvári [d,*], Hajieh K. Jabbari [d]

[a] *Edutus College, Tatabánya, Hungary*
[b] *Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary*
[c] *Department of Computer Science and Systems Technology, University of Pannonia, Veszprém, Hungary*
[d] *Department of Industrial Engineering, Eastern Mediterranean University, Turkey*

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A B S T R A C T

The main result of this paper is that the polytope of the bipartite TSP is significantly different from that of the general TSP. Comb inequalities are known as facet defining ones in the general case. In the bipartite case, however, many of them are satisfied whenever all degree and subtour elimination constraints are satisfied, *i.e.* these comb inequalities are not facet defining. The inequalities in question belong to the cases where vertices of one of the two classes occur in less than the half of the intersections of the teeth and the hand. Such side conditions are necessary, as simple example shows that the comb inequality can be violated when each class has vertices in more than the half of the intersections.

1. Introduction

The Traveling Salesman Problem (TSP) has plenty of applications in operations research. It also has a nice structure and is one of the most investigated NP-complete problems of combinatorial optimization; see *e.g.* the monographs [1,9].

Especially a lot of efforts have been made to explore the structure of the facets of its polytope. The most important facet defining inequalities are the degree constraints and the subtour elimination inequalities. The famous Dantzig–Fulkerson–Johnson (DFJ) model [6] contains only these constraints, and only the characteristic vectors of the complete tours satisfy the constraints among the integer vectors. However, it is well-known that the polytope of the DFJ model is strictly larger than the TSP polytope, *i.e.* the convex hull of the characteristic vectors of the tours. The next important class of facet defining inequalities is the set of the so-called comb inequalities. This type of constraints has several generalizations including clique-tree, domino, star, path, ladder and bipartite inequalities. Unfortunately, a complete description of the TSP polytope is not known [12]. Moreover, all of these results concern complete graphs. However, if the underlying graph is not K_n then the TSP polytope can be different.

Facet defining inequalities have great algorithmic importance. Branch and Bound, Branch and Cut, and Cut and Branch methods are used as general approach to solve TSP. All of them are based on the linear programming relaxation of TSP. If an optimal solution of the relaxation is not the characteristic vector of a tour then the bound can be improved by introducing a violated facet defining inequality. There are effective methods to find violated subtour elimination constraints depending on the size of the problem [10,11], and [13]. There are also some methods to find violated comb inequalities.

The underlying graph structure affects the difficulty of the particular TSP. An interesting problem class is bipartite TSP, which has application in controlling assembly robots. The robot moves from cells where parts are stored to an assembly

* Correspondence to: Department of IE, Eastern Mediterranean University, Famagusta, Mersin-10, Turkey.
E-mail addresses: kovacs.gergely@edutus.hu (G. Kovács), tuza@dcs.uni-pannon.hu (Z. Tuza), bela.vizvari@emu.edu.tr (B. Vizvári), hajieh.jabbari@cc.emu.edu.tr (H.K. Jabbari).

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point to fix the part and goes to another cell after fixing. Thus, its moves have a bipartite nature. The productivity depends on the length of the route of the robot arm, and therefore it has to be minimized.

The first paper on bipartite TSP is [7], followed only by rare further publications [2–4]. These papers concentrate on heuristic methods and do not discuss the issue of the TSP polytope. To the best of our knowledge, the present paper is the first one which has results on the special properties of the bipartite TSP polytope. Our main result is that a large part of the comb inequalities is not facet defining, although they are in the case of the complete graph.

This paper is organized as follows. The next section contains some basic notations and notions, also including the description of the comb inequalities. The main result is discussed in Section 3. The paper is finished with a short conclusion.

2. Basic notations and notions

TSP asks for a shortest Hamiltonian tour in an edge-weighted graph. In its original form the graph is the complete graph K_n. Many valid inequalities of the TSP polytope are generated by analyzing how many edges a Hamiltonian tour can have in a special graph structure. As a matter of fact, even the subtour elimination constraints can be discussed in that way. Indeed, for every integer m with $2 \leq m < n$, the subtour elimination constraint gives the upper bound $m-1$ for the number of edges a tour can have in a K_m subgraph of K_n. (The formal definition will be given below.)

The basic model of [6] uses the binary variables x_{ij} where

$$x_{ij} = \begin{cases} 1 & \text{if the tour goes directly between cities } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$

The number of variables x_{ij} is $\binom{n}{2}$ in the undirected case. Therefore the variables x_{ij} and x_{ji} are considered identical for notational convenience. Each city has two connections. Thus, the variables must satisfy the constraints

$$\sum_{j=1;\, j\neq i}^{n} x_{ij} = 2. \tag{1}$$

In what follows, it is never used that the degrees of vertices must be exactly 2; we only need that the degrees are at most 2, *i.e.*

$$\sum_{j=1;\, j\neq i}^{n} x_{ij} \leq 2. \tag{2}$$

For all $1 \leq i \leq n$, inequalities (2) are called *degree constraints*. They still allow the formation of small subtours, which are excluded by the so-called *subtour elimination constraints*:

$$\forall \mathcal{S} \subset \{1, 2, \ldots, n\},\ 3 \leq |\mathcal{S}| \leq n-1 :\ \sum_{i,j\in\mathcal{S}} x_{ij} \leq |\mathcal{S}| - 1. \tag{3}$$

Only the characteristic vectors of the tours are satisfying both (2) and (3) among the binary vectors. Continuous relaxation is obtained if the variables are allowed to take any value between 0 and 1, *i.e.* instead of $x_{ij} \in \{0, 1\}$ it is required only that

$$\forall\, 1 \leq i, j \leq n,\ i \neq j :\ 0 \leq x_{ij} \leq 1. \tag{4}$$

The relaxation is also called Linear Programming (LP) relaxation, especially if there is a linear objective function, because then the problem becomes an LP problem. In general, a linear programming problem is an LP relaxation of TSP if any set of valid inequalities (cuts) of the TSP polytope is a constraint set of the LP problem. However, the constraints (2)–(4) are easy ones, thus it makes no sense to omit any of them.

Let $\mathbf{x}$ be a feasible solution of the relaxed problem. Then the sum in (3) is denoted by $\mathbf{x}(\mathcal{S})$. Thus an equivalent form of (3) is

$$\forall \mathcal{S} \subset \{1, 2, \ldots, n\},\ 3 \leq |\mathcal{S}| \leq n-1 :\ \mathbf{x}(\mathcal{S}) \leq |\mathcal{S}| - 1. \tag{3'}$$

The comb inequalities were discovered by Chvátal [5]. They are the generalizations of Edmonds' 2-matching inequalities. The *comb* is a special graph structure consisting of two types of sets of vertices: a *hand* and an odd number of *teeth*. The intersection of the hand with any tooth must be non-empty. The graph contains only edges such that both vertices of the edge are in the same set, *i.e.* either in the hand or in one of the teeth. Let t be the number of teeth. It must be at least 3. In a more formalized way the comb must satisfy the following constraints. Let $\mathcal{H}$, $\mathcal{T}_i$ $(i = 1, \ldots, t)$ be the hand and the teeth, respectively. Then

$$\mathcal{H} \cap \mathcal{T}_i \neq \emptyset \quad \text{for} \quad i = 1, \ldots, t,$$

$$\forall\, 1 \leq i, j \leq t,\ i \neq j :\ \mathcal{T}_i \cap \mathcal{T}_j = \emptyset,$$

$$t \geq 3 \text{ and } t \text{ is odd.}$$