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Integer k-matchings of graphs☆

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ABSTRACT

An integer k-matching of a graph G is a function f that assigns to each edge an integer in $\{0, 1, \ldots, k\}$ such that $\sum_{e\in\Gamma(v)} f(e) \le k$ for each $v \in V(G)$. The k-matching number of G is the maximum number of $\sum_{e\in E(G)} f(e)$ over all k-matchings f. In this paper, when k is even, we give a relationship between some special fractional matchings and integer k-matchings, and then we obtain a formula for k-matching number by using fractional matching number and all the maximum integer k-matchings with the maximum number of edges assigned 0 (named 0-edges) can be constructed by using the algorithms given by Pulleyblank (1987) for generating some special fractional matchings. When k is odd, we obtain some properties of the maximum k-matchings with the maximum number of 0-edges.

1. Introduction and preliminaries

All graphs considered are finite and undirected. Let G be a graph, $v \in V(G)$ and $\Gamma(v)$ the set of edges incident with v. A *matching* of G is a subset of $E(G)$ in which no two edges are adjacent, equivalently, a matching is a function $x : E(G) \to \{0, 1\}$ such that $\sum_{e\in\Gamma(v)} x(e) \le 1$ for each vertex v. Clearly, $\sum_{e\in E(G)} x(e) \le \frac{|V(G)|}{2}$. The *matching number* of G, denoted by $\mu(G)$, is the maximum number of $\sum_{e\in E(G)} x(e)$ over all matchings x. A matching is *maximum* if $\sum_{e\in E(G)} x(e) = \mu(G)$. A matching x is *perfect* if $\sum_{e\in\Gamma(v)} x(e) = 1$ for each vertex v (i.e., v is x-saturated).

Fractional matching is a kind of relaxation of matching. A *fractional matching* of G is a function $g : E(G) \to [0, 1]$ such that $\sum_{e\in\Gamma(v)} g(e) \le 1$ for each vertex v. The *fractional matching number* of G, denoted by $\mu_f(G)$, is the supremum of $\sum_{e\in E(G)} g(e)$ over all fractional matchings g. A fractional matching g is maximum if $\sum_{e\in E(G)} g(e) = \mu_f(G)$. A vertex v of G is *saturated* by a fractional matching g or v is g-saturated if $\sum_{e\in\Gamma(v)} g(e) = 1$, otherwise, v is g-*unsaturated*.

Balinski [1] showed that a fractional matching g is a vertex of the polytope $\{f : f(e) \in [0, 1]$ for each edge e and $\sum_{e\in\Gamma(v)} f(e) \le 1$ for each vertex $v\}$ if and only if $g(e) \in \{0, \frac{1}{2}, 1\}$ for each $e \in E(G)$ and the edges e having $g(e) = \frac{1}{2}$(named $\frac{1}{2}$−edges) form vertex disjoint odd cycles of G. Such fractional matchings (which are a vertex of the polytope) are called *basic*. In fact, there exists a basic fractional matching for any graph ([9] See Theorem 2.1.5). If g is a basic fractional matching, then either $\sum_{e\in\Gamma(v)} g(e) = 1$ (i.e., v is g-saturated) or $\sum_{e\in\Gamma(v)} g(e) = 0$ for each vertex v of G. Let g be a fractional matching.

The *support* of a fractional matching g is the subset $S(g)$ of $E(G)$ consisting of all edges e having $g(e) \ne 0$. Then for a basic fractional matching g, each component of the subgraph induced by $S(g)$ is either a single edge (i.e., a pair of vertices joined by an edge) or an odd cycle. We say that the subset of edges e such that $g(e) = 1$ (named 1-edges) is the *integer part* of g, denoted by $I(g)$, and $S(g) - I(g)$ is the *fractional part* of g, denoted by $F(g)$.

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In 1965, Gallai and Edmonds [6] gave a decomposition of a graph according to maximum matchings, named Gallai–Edmonds Structure Theorem (see Proposition 1.1). Pulleyblank [8] defined a U-fractional matching and an $\mathcal{M}$-fractional matching. A *U-fractional matching* is a basic fractional matching g with the maximum number of saturated vertices such that $F(g)$ is minimal. An *$\mathcal{M}$-fractional matching* is a basic fractional matching with the maximum number of saturated vertices such that $|F(g)|$ is minimum. It is easy to see that every $\mathcal{M}$-fractional matching is a U-fractional matching. In [8], the structures of U-fractional matchings and $\mathcal{M}$-fractional matchings are characterized by using the Gallai–Edmonds Structure Theorem. Thus all $\mathcal{M}$-fractional matchings can be constructed by using the algorithms in [8].

Integer k-matching is a kind of generalization of matching. Let k be an integer. A *k-matching* of a graph G is a function $h : E(G) \to \{0, 1, 2, \ldots, k\}$ such that $\sum_{e\in\Gamma(v)} h(e) \le k$ for each vertex v. A vertex v of G is *saturated* by a k-matching h or v is h-saturated if $\sum_{e\in\Gamma(v)} h(e) = k$, otherwise, v is *h-unsaturated*. The *k-matching number* of G, denoted by $\mu_k(G)$, is the maximum number of $\sum_{e\in E(G)} h(e)$ over all k-matchings h. Then $\mu_k(G) \le \frac{k|V(G)|}{2}$. A k-matching h is *maximum* if $\sum_{e\in E(G)} h(e) = \mu_k(G)$. A k-matching h is *perfect* if every vertex is h-saturated. Then a k-matching h is perfect if and only if $\sum_{e\in E(G)} h(e) = \frac{k|V(G)|}{2}$.

In 2014, H. Lu and W. Wang [7] studied the perfect k-matching of general graphs and gave a sufficient and necessary condition for its existence. However, the problems about the structure of some special maximum k-matchings are open. In this paper, we characterize the structure of the maximum k-matchings with the maximum number of 0-edges. When k is even, we give a relationship between some special fractional matchings and integer k-matchings and then we obtain a formula for k-matching number by using fractional matching number and all the maximum integer k-matchings with the maximum number of 0-edges can be constructed by using the algorithms for generating special fractional matchings given by Pulleyblank [8]. When k is odd, the problem is open. Anyhow, we characterize the maximum k-matchings with the maximum number of 0-edges and obtain some results.

In the following, we introduce the Gallai–Edmonds decomposition. Let $D(G)$ be the set of vertices of G which are missed by at least one maximum matching of G, and $A(G)$ the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$. Finally, let $C(G) = V(G) - A(G) - D(G)$. A graph G is said to be *factor-critical* if $G - v$ has a perfect matching for any vertex $v \in V(G)$. A matching is said to be a *near-perfect matching* if it covers all vertices but one. The number of components of a graph G is denoted by $c(G)$. The subgraph of G induced by a vertex subset S is denoted by $\langle S\rangle$.

Proposition 1.1 (*[6] Gallai–Edmonds Structure Theorem*)**.** *Let $D(G)$, $A(G)$, and $C(G)$ be defined as above. Then*

(1) *Every component of $\langle D(G)\rangle$ is factor-critical.*

(2) *The subgraph $\langle C(G)\rangle$ has a perfect matching.*

(3) *A matching of G is maximum if and only if it consists of a near-perfect matching of each component of $\langle D(G)\rangle$, a perfect matching of $\langle C(G)\rangle$, and a matching which matches every vertex in $A(G)$ to one of distinct components of $D(G)$.*

(4) $\mu(G) = \frac{1}{2}[|V(G)| - c(\langle D(G)\rangle) + |A(G)|]$.

For a maximum matching M and a component G_i of $\langle D(G)\rangle$, we say that G_i is *M-full* if some vertex of G_i is matched with a vertex in $A(G)$, that is, every vertex of G_i is *M-saturated*, otherwise, G_i is *M-near full*. The number of nontrivial M-near full components is denoted by $nc(M)$. Let $nc(G) = max\{nc(M) \mid M \text{ is a maximum matching}\}$.

Proposition 1.2 (*[4]*)**.** *For any graph G,*

$$\mu_f(G) = \mu(G) + \frac{nc(G)}{2}.$$

Now, we study the condition such that $nc(M) = nc(G)$ for a maximum matching M. Let D_0 be the set of vertices in $D(G)$ which form trivial components of $\langle D(G)\rangle$ and $N(D_0)$ the neighbor set of D_0. Then $N(D_0) \subseteq A(G)$. Let $D_0(M) = \{v \in D_0 \mid v$ is an M-near full component of $\langle D(G)\rangle\}$. Then $c(\langle D(G)\rangle) = |A(G)| + |D_0(M)| + nc(M)$ since the number of M-full components of $\langle D(G)\rangle$ is $|A(G)|$. Thus $nc(M)$ is maximum if and only if $|D_0(M)|$ is minimum, equivalently, $|D_0 - D_0(M)|$ (which is the number of M-saturated vertices in D_0) is maximum. So it implies the following proposition.

Proposition 1.3. *Let G be a graph, M a maximum matching of G and D_0 defined as above. Then $nc(M) = nc(G)$ if and only if M induces a maximum matching of $\langle D_0 \bigcup N(D_0)\rangle$.*

2. Some results on fractional matchings

Recently, some results about the fractional matching number are obtained (see [2,3,10]). In this section, we study some special maximum fractional matchings which are useful for studying k-matchings. A maximum fractional matching g is said to be *H-fractional matching* if g has the maximum number of 0-edges.

Lemma 2.1 (*[5]*)**.** *For any graph G, any H-fractional matching of G is basic.*

Lemma 2.2 (*[8]*)**.** *Let G be a graph, $(D(G), A(G), C(G))$ the Gallai–Edmonds partition of G and g an $\mathcal{M}$-fractional matching. Then*

(1) *g induces a perfect matching of $\langle C(G)\rangle$.*

(2) *For each $u \in A(G)$, there exists $v \in D(G)$ adjacent to u such that $g(uv) = 1$.*