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Dynamic coloring parameters for graphs with given genus

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ABSTRACT

A proper vertex coloring of a graph G is *r-dynamic* if for each $v \in V(G)$, at least $\min\{r, d(v)\}$ colors appear in $N_G(v)$. In this paper we investigate r-dynamic versions of coloring, list coloring, and paintability. We prove that planar and toroidal graphs are 3-dynamically 10-colorable, and this bound is sharp for toroidal graphs. We also give bounds on the minimum number of colors needed for any r in terms of the genus of the graph: for sufficiently large r, every graph with genus g is r-dynamically $((r+1)(g+5)+3)$-colorable when $g \leq 2$ and r-dynamically $((r+1)(2g+2)+3)$-colorable when $g \geq 3$. Furthermore, each of these upper bounds for r-dynamic k-colorability also holds for r-dynamic k-choosability and for r-dynamic k-paintability.

1. Introduction

For a graph G and positive integer r, an *r-dynamic coloring* of G is a proper vertex coloring such that for each $v \in V(G)$, at least $\min\{r, d(v)\}$ distinct colors appear in $N_G(v)$. The *r-dynamic chromatic number*, denoted $\chi_r(G)$, is the minimum k such that G admits an r-dynamic k-coloring. Montgomery [16] introduced 2-dynamic coloring and the generalization to r-dynamic coloring.

List coloring was introduced independently by Vizing [20] and by Erdős, Rubin, and Taylor [5]. A *list assignment* L for G assigns to each vertex v a list $L(v)$ of permissible colors. Given a list assignment L for a graph G, if a proper coloring ϕ can be chosen so that $\phi(v) \in L(v)$ for all $v \in V(G)$, then G is *L-colorable*. The *choosability* of G is the least k such that G is L-colorable for any list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$. We consider the r-dynamic version of this parameter. For further work, see [1,10,11]. A graph G is *r-dynamically L-colorable* when an r-dynamic coloring can be chosen from the list assignment L. The *r-dynamic choosability* of G, denoted $\mathrm{ch}_r(G)$, is the least k such that G is *r-dynamically L-colorable* for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$.

Zhu [23] and Schauz [17] independently introduced an online version of choosability, which is modeled by the following game.

Definition 1.1. Suppose G is a graph and that each vertex $v \in V(G)$ is assigned a positive number $f(v)$ of *tokens*. The *f-paintability game* is played by two players: Lister and Painter. On the ith round, Lister *marks* a nonempty set of uncolored vertices; each marked vertex loses one token. Painter responds by choosing a subset of the marked set that forms an independent set in the graph and assigning color i to each vertex in that subset. Lister wins the game by marking a vertex with no tokens, and Painter wins by coloring all vertices.

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We say G is *f-paintable* when Painter has a winning strategy in the f-paintability game. When G is f-paintable and $f(v) = k$ for all $v \in V(G)$, we say that G is *k-paintable*. The least k such that G is k-paintable is the *paint number* (or *online choice number*) of G, denoted by $\mathring{\text{ch}}(G)$.

In the f-paintability game, Painter's goal is to generate a proper coloring of the graph. We say that a graph G is *r-dynamically k-paintable* when Painter has a winning strategy that produces an r-dynamic coloring of G when all vertices have k tokens. The least k such that Painter can accomplish this is the *r-dynamic paint number*, denoted by $\mathring{\text{ch}}_r(G)$.

The *square* of a graph G, denoted G^2, is the graph resulting from adding an edge between every pair of vertices of distance 2 in G. For any graph G, it is clear that

$$\begin{array}{ccccccccccc} \chi(G) & = & \chi_1(G) & \le & \chi_2(G) & \le & \cdots & \le & \chi_{\Delta(G)}(G) & = & \cdots & = & \chi(G^2), \\ \text{ch}(G) & = & \text{ch}_1(G) & \le & \text{ch}_2(G) & \le & \cdots & \le & \text{ch}_{\Delta(G)}(G) & = & \cdots & = & \text{ch}(G^2), \\ \mathring{\text{ch}}(G) & = & \mathring{\text{ch}}_1(G) & \le & \mathring{\text{ch}}_2(G) & \le & \cdots & \le & \mathring{\text{ch}}_{\Delta(G)}(G) & = & \cdots & = & \mathring{\text{ch}}(G^2), \end{array} \tag{1}$$

and that $\chi_r(G) \le \text{ch}_r(G) \le \mathring{\text{ch}}_r(G)$ for all r. Thus we can think of r-dynamic coloring as bridging the gap between coloring a graph and coloring its square.

Wegner [22] conjectured bounds for the chromatic number of squares of planar graphs given their maximum degree. For a graph G with $\Delta(G) \le 3$, proper colorings of G^2 and 3-dynamic colorings of G are equivalent. Wegner showed that $\chi_3(G) \le 8$ for any planar subcubic graph G, but it is an open question whether this is sharp. (Some have reported that Thomassen proved $\chi_3(G) \le 7$ for such graphs, but apparently the proof was flawed and the problem remains open [14].) Cranston and Kim [4] proved that when G is a planar subcubic graph, $\text{ch}_3(G) \le 7$ if the girth is at least 7 and $\text{ch}_3(G) \le 6$ if the girth is at least 9.

Thomassen [19] proved that planar graphs are 5-choosable, and Voigt [21] proved sharpness. Schauz [17] further proved that planar graphs are 5-paintable. Kim, Lee, and Park [12] proved that planar graphs are actually 2-dynamically 5-choosable. Their proof involves showing that every planar graph has a planar supergraph with an edge in the neighborhood of every vertex. They then invoke Thomassen's result that planar graphs are 5-choosable to obtain their result. By using Schauz's result that planar graphs are 5-paintable instead, Kim, Lee, and Park's result is strengthened to the following corollary.

Corollary 1.2. *If G is a planar graph, then $\mathring{ch}_2(G) \le 5$.*

Heawood [6] proved that for $g > 0$, graphs of (orientable) genus g are $(h(g) - 1)$-degenerate and hence $h(g)$-colorable, where

$$h(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

Because $(k-1)$-degenerate graphs are k-paintable, this also shows that graphs with genus g are $h(g)$-paintable. Chen et al. [3] proved that such a graph is 2-dynamically $h(g)$-choosable. Mahoney [15] strengthened their result to prove that such a graph is 2-dynamically $h(g)$-paintable.

Our main results are on the 3-dynamic chromatic number, choosability, and paint numbers for planar and toroidal graphs. We will call a graph *toroidal* if it can be drawn on the torus without crossing edges; in particular, we consider planar graphs to be also toroidal.

Theorem 1.3. *If G is a toroidal graph, then $\chi_3(G) \le \text{ch}_3(G) \le \mathring{\text{ch}}_3(G) \le 10$.*

Theorem 1.3 is sharp: the Petersen graph P has maximum degree 3 and diameter 2, so $\chi_3(P) = \chi(P^2) = \chi(K_{10}) = 10$.

Our proofs use the Discharging Method. A *configuration* in a graph is a set of vertices that satisfies some specified condition, for example, a condition on the degrees or adjacencies of the vertices in the configuration. We say that a configuration in a graph is *reducible* for a graph property if it cannot occur in a minimal graph failing that property. We say that a partial coloring of a graph G *extends* if the uncolored vertices can be assigned colors so that the coloring for all of $V(G)$ is an r-dynamic coloring of G. We obtain an immediate corollary of Theorem 1.3.

Corollary 1.4. *If G is a planar graph, then $\chi_3(G) \le \text{ch}_3(G) \le \mathring{\text{ch}}_3(G) \le 10$.*

We do not believe that Corollary 1.4 is sharp; the proof we give relies heavily on showing that the configuration consisting of two adjacent vertices both having degree 3 is reducible for 3-dynamic 10-paintability. An example of a planar graph G with $\chi_3(G) = 7$ is the graph obtained from K_4 by subdividing the three edges incident to one vertex, shown in Fig. 1. Note that G has maximum degree 3 and diameter 2, so $\chi_3(G) = \chi(G^2) = \chi(K_7)$.

In Section 3, we show that several configurations are reducible for 3-dynamic 10-paintability, which also implies reducibility for 3-dynamic 10-choosability and 3-dynamic 10-colorability. In Section 4, we complete the proof of Theorem 1.3 by using the Discharging Method to show that the configurations listed in Section 3 form a set that is unavoidable by a toroidal graph.

Finally, in Section 5, we consider graphs with higher genus. Let $\gamma(G)$ denote the minimum genus of a surface on which G embeds.

Theorem 1.5. *Let G be a graph, and let $g = \gamma(G)$.*