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On strong edge-coloring of graphs with maximum degree 4

Jian-Bo Lv [a], Xiangwen Li [a,*], Gexin Yu [a,b]

[a] School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PR China
[b] Department of Mathematics, The College of William and Mary, Williamsburg, VA, 23185, USA

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ABSTRACT

The strong chromatic index of a graph G, denoted by $\chi_s'(G)$, is the least number of colors needed to edge-color G properly so that every path of length 3 uses three different colors. In this paper, we prove that if G is a graph with $\Delta(G) = 4$ and maximum average degree less than $\frac{61}{18}$ (resp. $\frac{7}{2}$, $\frac{18}{5}$, $\frac{15}{4}$, $\frac{51}{13}$), then $\chi_s'(G) \le 16$ (resp. 17, 18, 19, 20), which improves the results of Bensmail et al. (2015).

1. Introduction

A *strong edge-coloring* of a graph G is a proper edge-coloring of G such that the edges of any path of length 3 use three different colors. It follows that each color class of a strong edge-coloring is an induced matching. The strong chromatic index of a graph G, denoted by $\chi_s'(G)$, is the smallest integer k such that G can be strongly edge-colored with k colors. The concept of strong edge-coloring was introduced by Fouquet and Jolivet in [8,9] and can be used to model conflict-free channel assignment in radio networks in [16,17].

In 1985, Erdős and Nešetřil proposed the following interesting conjecture.

Conjecture 1.1 ([7]). *For a graph G with maximum degree Δ,*

$$\chi_s'(G) \le \begin{cases} \dfrac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \dfrac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

When $\Delta \le 3$, Conjecture 1.1 has been verified by Andersen [1], and independently by Horák, Qing, and Trotter [13]. When Δ is sufficiently large, Molloy and Reed in [15] proved that $\chi_s'(G) \le 1.998\Delta(G)^2$, using probabilistic techniques. This bound is improved to $1.93\Delta^2$ by Bruhn and Joos [4], and very recently, is further improved to $1.835\Delta^2$ by Bonamy, Perrett, and Postle [3].

The maximum average degree of a graph G, $mad(G)$, is defined to be the maximum average degree over all subgraphs of G. Hocquard et al. [11,12] and DeOrsey et al. [6] studied the strong chromatic index of subcubic graphs with bounded maximum average degree.

* Corresponding author.
E-mail address: xwli68@mail.ccnu.edu.cn (X. Li).

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We study graphs with maximum degree 4, which are conjectured to be colorable with at most 20 colors in Conjecture 1.1. Cranston [5] showed that 22 colors suffice, which is improved to 21 colors very recently by Huang, Santana and the third author [14]. However, it is still not clear if 20 colors suffice even if the minimum degree of such graphs is 3. Bensmail, Bonamy, and Hocquard [2] studied the strong chromic index of graphs with maximum degree four and bounded maximum average degrees.

Theorem 1.2 (*Bensmail, Bonamy, and Hocquard [2]*). *For every graph G with $\Delta = 4$,*

(1) *If $mad(G) < \frac{16}{5}$, then $\chi_s'(G) \le 16$.*
(2) *If $mad(G) < \frac{10}{3}$, then $\chi_s'(G) \le 17$.*
(3) *If $mad(G) < \frac{17}{5}$, then $\chi_s'(G) \le 18$.*
(4) *If $mad(G) < \frac{18}{5}$, then $\chi_s'(G) \le 19$.*
(5) *If $mad(G) < \frac{19}{5}$, then $\chi_s'(G) \le 20$.*

In this paper, we improve the results from [2] as follows.

Theorem 1.3. *For every graph G with $\Delta = 4$, each of the following holds.*

(1) *If $mad(G) < \frac{61}{18}$, then $\chi_s'(G) \le 16$.*
(2) *If $mad(G) < \frac{7}{2}$, then $\chi_s'(G) \le 17$.*
(3) *If $mad(G) < \frac{18}{5}$, then $\chi_s'(G) \le 18$.*
(4) *If $mad(G) < \frac{15}{4}$, then $\chi_s'(G) \le 19$.*
(5) *If $mad(G) < \frac{51}{13}$, then $\chi_s'(G) \le 20$.*

From the proof of Theorem 1.3(5), we obtain the following corollary, which implies Conjecture 1.1 is true in some spacial cases.

Corollary 1.4. *For every graph G with $\Delta = 4$, if there are two 3-vertices whose distance is at most 4, then $\chi_s'(G) \le 20$.*

We end this section with notation and terminology. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $d_G(v)$ denote the degree of a vertex v in a graph G. We use V, E and $d(v)$ for $V(G)$, $E(G)$ and $d_G(v)$, respectively, if it is understood from the context. Denote by $d(u, v)$ the distance between vertices u and v of G. A vertex is a *k-vertex* (*k^--vertex*) if it is of degree k (at most k). Similarly, a neighbor of a vertex v is a *k-neighbor* of v if it is of degree k. A 4-vertex is *special* if it is adjacent to a 2-vertex. A 3-vertex is a *3_k-vertex* if it is adjacent to k 3-vertices, where $k = 0, 1, 2$. A *4_k-vertex* is a 4-vertex adjacent to exactly k 3-vertices. Denote by $N(v)$ the neighborhood of the vertex v, let $N_i(v) = \{u \in V(G) : d(u, v) = i\}$ for $i \ge 1$. For simplicity, $N_0(v) = \{v\}$ and $N_1(v) = N(v)$. Let $L_i(v) = \cup_{j=0}^{i} N_j(v)$ and $D_3(G) = \{v \in V(G) : d(v) = 3\}$. For a graph $G = (V, E)$ and $E' \subseteq E$, G has a *partial edge-coloring* if $G[E']$ has a strong edge-coloring, where $G[E']$ is the graph with vertex set V and edge set E'.

In the proof of Theorem 1.3, the well known result of Hall [10] is applied in terms of systems of distinct representatives.

Theorem 1.5 (*[10]*). *Let $A_1, \ldots, A_n$ be n subsets of a set U. A system of distinct representatives of $\{A_1, \ldots, A_n\}$ exists if and only if for all k, $1 \le k \le n$ and every subcollection of size k, $\{A_{i_1}, \ldots, A_{i_k}\}$, we have $|A_{i_1} \cup \cdots \cup A_{i_k}| \ge k$.*

2. Proof of Theorem 1.3

Let H be a counterexample to Theorem 1.3 with $|V(H)| + |E(H)|$ minimized. That is, for some

$$(m, k) \in \{(\frac{61}{18}, 16), (\frac{7}{2}, 17), (\frac{18}{5}, 18), (\frac{15}{4}, 19), (\frac{51}{13}, 20)\}$$

we have $mad(H) < m$ and $\chi_s'(H) > k$.

By the minimality of H, $\chi_s'(H - e) \le k$ for each $e \in E(H)$, and we may assume that H is connected. Denote by $[k] = \{1, 2, \ldots, k\}$ the set of colors. If $e = uv$ is an uncolored edge in a partial coloring of H, then let $L_H(e)$ be the set of colors that is used on the edges incident to a vertex in $N_H(u) \cup N_H(v)$, and let $L_H'(e) = [k] \setminus L_H(e)$. We write $L(e)$ and $L'(e)$ for $L_H(e)$ and $L_H'(e)$, respectively, if it is clear from the context. We now establish some properties of H.

Lemma 2.1. *Let x be a vertex of H with $d(x) = d$. If the edges incident to x can be ordered as $xy_1, xy_2, \ldots, xy_d$ such that in a partial k-coloring of $H - x$, $|L(xy_i)| \le k - i$, then the partial coloring can be extended to H. In particular,*

(a) *There is no 1-vertex in H, and if $k \ge 17$, then there is no 2-vertex in H.*
(b) *Each 2-vertex x in H has two 4-neighbors, each of which is adjacent to three 4-vertices.*

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