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The fault-diameter and wide-diameter of twisted hypercubes

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ABSTRACT

The ℓ-fault-diameter of a graph G is the minimum d such that the diameter of $G-X$ is at most d for any subset X of $V(G)$ of size $|X| \leq \ell-1$. The ℓ-wide-diameter of G is the minimum integer d for which there exist at least ℓ internally vertex disjoint paths of length at most d between any two distinct vertices in G. These two parameters measure the fault tolerance and transmission delay in communication networks modelled by graphs. Twisted hypercubes are variation of hypercubes defined recursively: K_1 is the only 0-dimension twisted hypercube, and for $n \geq 1$, an n-dimensional twisted hypercube G_n is obtained from the disjoint union of two $(n-1)$-dimensional twisted hypercubes G'_{n-1} and G''_{n-1} by adding a perfect matching between $V(G'_{n-1})$ and $V(G''_{n-1})$. Recently, two types of twisted hypercubes $Z_{n,k}$ and H_n are introduced in Zhu (2017). This paper gives an upper bound for the n-fault-diameters of special families of twisted hypercubes of dimension n. As a consequence of this result, the n-fault-diameter of H_n is $(1+o(1))\frac{n}{\log_2 n}$. For $k = \lceil \log_2 n - 2\log_2\log_2 n \rceil$, the n-fault-diameter of $Z_{n,k}$ is also $(1+o(1))\frac{n}{\log_2 n}$. This bound is asymptotically optimal, as any n-dimensional twisted hypercube has diameter greater than $\frac{n}{\log_2 n}$. Then we extend a result in Qi and Zhu (2017) about $Z_{n,k}$ to H_n and prove that the $\kappa(n)$-wide-diameter of H_n is $(1+o(1))\frac{n}{\log_2 n}$.

1. Introduction

Graphs are used as model for interconnection networks. The vertices correspond to processors and the edges correspond to links between processors. The diameter of a graph G is an important parameter in the study of networks. It measures the communication delay of the network. For a communication network to be reliable, one requires that a network has small diameter even if some vertices are removed. This motivated the definition of fault-diameter, introduced by Krishnamoorthy and Krishnamurthy in [10] in 1987.

Definition 1.1. Assume G is ω-connected and $\ell \leq \omega$. The *ℓ-fault-distance* between x and y, denoted by $d^f_\ell(x,y)$, is defined as $d^f_\ell(x,y) = \max\{d_{G-X}(x,y) : X \subseteq V(G) - \{x,y\}, |X| \leq \ell - 1\}$. Here $d_{G-X}(x,y)$ is the distance between x and y in the graph $G-X$. The *ℓ-fault-diameter* of G, denoted by $d^f_\ell(G)$, is the maximum value $d^f_\ell(x,y)$ among all pairs (x,y) of vertices of G.

The wide-diameter of a graph is another parameter related to the reliability of the network, and to the efficiency of transmitting large amount of data through disjoint routes. This parameter was introduced independently by Hsu and Lyuu [9] and Flandrin and Li [6].

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Definition 1.2. Assume G is ω-connected and $\ell \le \omega$. The *ℓ-wide-distance* between x and y, denoted by $d_\ell(x, y)$, is the smallest integer d such that there are ℓ internally vertex disjoint x-y-paths of lengths at most d. The *ℓ-wide-diameter* of G, denoted by $d_\ell(G)$, is the maximum value $d_\ell(x, y)$ among all pairs (x, y) of vertices of G.

Observe that both the 1-wide-diameter of G and the 1-fault-diameter of G are the same as the diameter of G. If G is ω-connected and $\ell \le \omega$, then $d(G) \le d^f_\ell(G) \le d_\ell(G)$ and $d(G) = d_1(G) \le \cdots \le d_{\ell-1}(G) \le d_\ell(G)$.

Hypercubes and its variations are popular models for communication networks. Hypercube Q_n has vertex set Z_2^n (i.e., its vertices are binary strings of length n) in which two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are adjacent if and only if x and y differ in exactly one bit, i.e., there is an index i such that $x_i \ne y_i$ and for $j \ne i$, $x_j = y_j$. Alternatively, hypercubes can be defined recursively: $Q_0 = K_1$ consists of a single vertex, and for $n \ge 1$, Q_n is obtained from two copies of Q_{n-1}, say $0Q_{n-1}$ and $1Q_{n-1}$, by adding edges connecting $0x$ to $1x$ for each vertex x of Q_{n-1}. Twisted hypercubes are variations of hypercubes, also defined recursively.

Definition 1.3. For $n \ge 1$, a *twisted hypercube* of dimension n is obtained from two twisted hypercubes of dimension $n-1$ by adding a perfect matching joining the vertex sets of the two $(n-1)$-dimension twisted hypercubes. The unique twisted hypercube of dimension 0 consists of one vertex and no edges.

By definition, there is only one 0-dimensional twisted hypercube, and it is also easy to verify that for $n \le 2$, up to isomorphism, there is only one n-dimensional twisted hypercube. However, for $n \ge 3$, there are more than one n-dimensional twisted hypercubes.

For convenience, we assume that all n-dimensional twisted hypercubes have vertex set Z_2^n, which is the set of binary strings of length n.

If x is a binary string of length n_1 and y is a binary string of length n_2, then xy is the *concatenation* of x and y, which is a binary string of length $n_1 + n_2$. If Z is a set of binary strings, then let $xZ = \{xy : y \in Z\}$. So $Z_2^n = 0Z_2^{n-1} \cup 1Z_2^{n-1}$.

Definition 1.4. Assume G_{n-1} and G'_{n-1} are $(n-1)$-dimensional twisted hypercubes, and π is a permutation of Z_2^n of order 2 such that $\pi(0Z_2^{n-1}) = 1Z_2^{n-1}$. We write $G_n = T(G_{n-1}, G'_{n-1}, \pi)$ to mean that G_n is the n-dimensional twisted hypercube with vertex set Z_2^n, in which the subset $0Z_2^{n-1}$ induces a copy of G_{n-1} and the subset $1Z_2^{n-1}$ induces a copy of G'_{n-1} and for each $x \in Z_2^n$, there is an edge connecting x to $\pi(x)$.

It follows from the definition that each n-dimensional twisted hypercube G_n is of the form $G_n = T(G_{n-1}, G'_{n-1}, \pi)$ for some $(n-1)$-dimensional twisted hypercubes G_{n-1} and G'_{n-1} and a permutation π of Z_2^n of order 2 such that $\pi(0Z_2^{n-1}) = 1Z_2^{n-1}$.

Definition 1.5. A family $\mathcal{H}$ of twisted hypercubes is called *closed* if for any twisted n-dimensional hypercube $G_n \in \mathcal{H}$, there are $(n-1)$-dimensional hypercubes $G_{n-1}, G'_{n-1} \in \mathcal{H}$ such that $G_n = T(G_{n-1}, G'_{n-1}, \pi)$ for a permutation π of Z_2^n of order 2 such that $\pi(0Z_2^{n-1}) = 1Z_2^{n-1}$.

Some closed families of twisted hypercubes such as cross cube, Möbius cube, spined cube, and some other unnamed families of twisted hypercubes have been studied in the literature [1–3,5,7,12,13].

An n-dimensional twisted hypercube is an n-regular graph with 2^n vertices. So its diameter is larger than $n/\log_2 n$. It is known that the hypercube of dimension n has diameter n, and the n-dimensional cross cube, Möbius cube has diameter about $n/2$, and the n-dimensional spined cube has diameter about $n/3$. Hartman [8] provided an explicit inductive construction of an n-dimensional twisted hypercube with diameter at most $cn/\log_2 n$, where c can be any number greater than 8 (this is the earliest result we know on the general model, which had been proposed in 1997 by Peter Slater). Recently, two types of twisted hypercubes, $Z_{n,k}$ and H_n, were introduced in [14]. It is shown in [14] that, for each fixed integer k, $Z_{n,k}$ has diameter at most $n/(k+1)+2^k$. If $k = \lceil \log_2 n - 2\log_2\log_2 n \rceil$, then this bound is $(1+o(1))\frac{n}{\log_2 n}$. The graph H_n also has diameter $(1+o(1))\frac{n}{\log_2 n}$. As any twisted hypercube of dimension n has diameter greater than $n/\log_2 n$, so H_n and $Z_{n,k}$ for $k = \lceil \log_2 n - 2\log_2\log_2 n \rceil$ have asymptotically optimal diameter among all twisted hypercubes.

In Section 2, we study the fault-diameter of graphs in closed families of twisted hypercubes. Assume $\mathcal{H}$ is a closed family of twisted hypercubes. If $f(n)$ is a non-decreasing function such that $d(G_n) \le f(n)$ for any n-dimensional twisted hypercube $G_n \in \mathcal{H}$, then for any n-dimensional twisted hypercube $G_n \in \mathcal{H}$, $d^f_n(G_n) \le f(n) + 2\lceil \log_2 n \rceil$. This implies that the n-fault-diameter of H_n is of order $(1+o(1))\frac{n}{\log_2 n}$, and for $k = \lceil \log_2 n - 2\log_2\log_2 n \rceil$, the n-fault-diameter of $Z_{n,k}$ is also of order $(1+o(1))\frac{n}{\log_2 n}$. In Section 3, we extend a result in [11] about the wide-diameter of graphs $Z_{n,k}$ to H_n, and prove that for $\kappa(n) = \lceil \log_2 n - 2\log_2\log_2 n \rceil$, the $\kappa(n)$-wide-diameter of H_n is $(1+o(1))\frac{n}{\log_2 n}$.

2. The fault-diameter of twisted hypercubes

We denote by $(0)_q$ (respectively, $(1)_q$) the binary string of length q with all the entries 0 (respectively, with all entries 1).

If P and P' are paths in G, and the last vertex of P is adjacent to the first vertex of P', then we denote by $P \vee P'$ the *concatenation* of P and P', which is the walk obtained by adding P' to the end of P. If P and P' have no vertex in common, then $P \vee P'$ is a path in G.

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