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A coloring algorithm for $4K_1$-free line graphs

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ABSTRACT

Given a family $\mathcal{F}$ of graphs, let *Free*($\mathcal{F}$) be the class of graphs that do not contain any member of $\mathcal{F}$ as an induced subgraph. When $\mathcal{F}$ is a set of four-vertex graphs the complexity of the vertex coloring problem in *Free*($\mathcal{F}$) is known, with three exceptions: $\mathcal{F} = \{\text{claw}, 4K_1\}$, $\mathcal{F} = \{\text{claw}, 4K_1, \text{co-diamond}\}$, and $\mathcal{F} = \{C_4, 4K_1\}$. In this paper, we study the coloring problem for *Free*(claw, $4K_1$). We solve the vertex coloring problem for a subclass of *Free*(claw, $4K_1$) which contains the class of $4K_1$-free line graphs.

1. Introduction

Graph coloring is one of the most important problems in graph theory and computer science. Given an integer k, a *k-coloring* of a graph G is a mapping $f : V(G) \to \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$. The *chromatic number*, $\chi(G)$, of G is the smallest integer k such that G admits a k-coloring. Let VERTEX COLORING be the problem of determining the chromatic number of a graph. This problem is NP-hard [10,17]. However, for some graph families the problem can be solved in polynomial time. Let $\mathcal{F}$ be a set of graphs. We say that a graph is $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as an induced subgraph, and we denote by *Free*($\mathcal{F}$) the class of $\mathcal{F}$-free graphs.

For a single graph H, it is proved in [19] that VERTEX COLORING in *Free*(H) is polynomial-time solvable if H is an induced subgraph of the P_4 or $P_3 + P_1$ and is NP-complete for any other graph H. This result motivates us to consider VERTEX COLORING in *Free*($\mathcal{F}$) when $\mathcal{F}$ is any family of four-vertex graphs. As this paper was being written, we discovered that Lozin and Malyshev [20] have considered the same problem. They determined the computational complexity of VERTEX COLORING in $\mathcal{F}$-free graphs for every finite set $\mathcal{F}$ that consists of four-vertex graphs, with four exceptions (see also [11]). The class Free($K_{1,3}$, $4K_1$) is among these classes and it is regarded as an important open problem. We found some results already discovered in [20], but we also found a new result which we will present in this paper. Let us first recall some notation. For vertex-disjoint graphs G and H, let $G + H$ denote the *disjoint union* of G and H, which is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. For a graph G and an integer k, let kG denotes the disjoint union of k copies of G. Let P_n (respectively, C_n, K_n) denote the chordless path (respectively, chordless cycle, clique) on n vertices. Let $K_n \setminus e$ denote the complete graph on n vertices minus one edge. Let *co-H* denote the complement of a graph H. Fig. 1 shows all graphs on four vertices with their names. The purpose of this paper is to prove the following theorem.

Theorem 1.1. VERTEX COLORING *is polynomial-time solvable in Free(claw,* $4K_1$, $K_5 \setminus e$*).*

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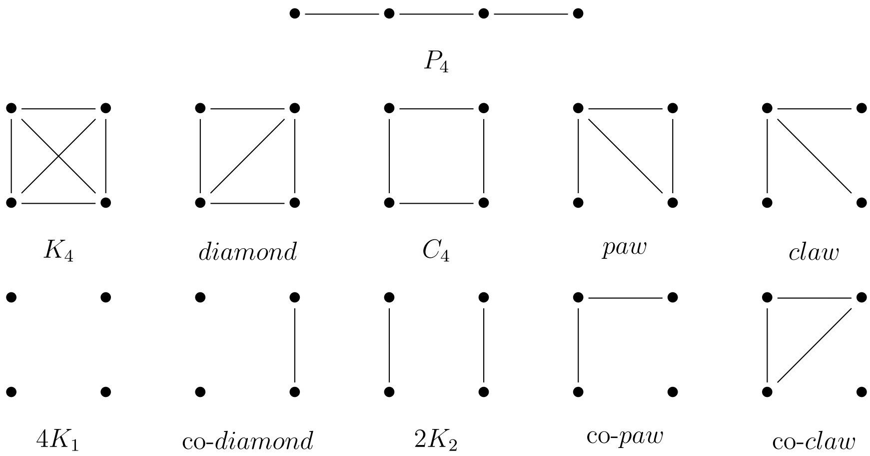

Fig. 1. All four-vertex graphs.

To explain this result, we need to discuss the background of the problem. Recall the following result from [19]:

Theorem 1.2 ([19]). *For a single graph H,* VERTEX COLORING *is polynomial-time solvable in the class Free(H) if H is an induced subgraph of P_4 or $P_3 + P_1$ and is NP-complete otherwise.*

The *clique-width* [8] of a graph is the minimum number of labels needed to build the graph using only the following four operations: (i) create a vertex u labeled by integer ℓ; (ii) make the disjoint union of several graphs; (iii) for some pair of distinct labels i and j, add all edges between vertices with label i and vertices with label j; (iv) for some pair of distinct labels i and j, relabel all vertices of label i by label j. In [25], the following result is established.

Theorem 1.3 ([25]). *For any constant c,* VERTEX COLORING *is polynomial-time solvable in the class of graphs with clique-width at most c.*

The clique-width of graphs in the class $Free(\mathcal{F})$ is studied in [4] when $\mathcal{F}$ is any family of four-vertex graphs: for some instances of $\mathcal{F}$ the clique-width is bounded and for all others it is unbounded. In the bounded case Theorem 1.3 implies that VERTEX COLORING is polynomial-time solvable. This severely reduces the number of remaining cases that must be considered. It is shown in [4] that there are exactly seven minimal classes with unbounded clique-width. These are:

$\mathcal{X}_1 = Free(\text{claw}, C_4, K_4, \text{diamond})$.
$\mathcal{X}_2 = Free(\text{co-claw}, 2K_2, 4K_1, \text{co-diamond})$.
$\mathcal{X}_3 = Free(C_4, \text{co-claw}, \text{paw}, \text{diamond}, K_4)$.
$\mathcal{X}_4 = Free(2K_2, \text{claw}, \text{co-paw}, \text{co-diamond}, 4K_1)$.
$\mathcal{X}_5 = Free(K_4, 2K_2)$.
$\mathcal{X}_6 = Free(C_4, 2K_2)$.
$\mathcal{X}_7 = Free(C_4, 4K_1)$.

Thus, if $\mathcal{F}$ is a set of four-vertex graphs and $Free(\mathcal{F}) \not\supseteq \mathcal{X}_i$ $(i = 1, 2, \ldots, 7)$, then VERTEX COLORING is polynomial-time solvable for $Free(\mathcal{F})$. We remark that VERTEX COLORING is NP-complete for:

- $\mathcal{X}_1$, due to a result in [19] which shows the problem is NP-complete for $Free(\text{claw}, C_4)$ and for $Free(\text{claw}, K_4, \text{diamond})$;
- $\mathcal{X}_2$, due to Theorem 6 in [26];
- $\mathcal{X}_3$, due to a remark in [19, Case 1 in Section 4], where it is shown that the problem is NP-complete for $Free(\mathcal{F})$ if every graph in $\mathcal{F}$ contains a cycle.

Thus, VERTEX COLORING is NP-complete for $Free(\mathcal{F})$ whenever $Free(\mathcal{F}) \supseteq \mathcal{X}_i$ for any $i \in \{1, 2, 3\}$. Therefore, we need only examine the problem for classes $\mathcal{X}_4$, $\mathcal{X}_5$, $\mathcal{X}_6$, $\mathcal{X}_7$ and their superclasses defined by forbidding induced subgraphs with four vertices. We note that:

- In [20], a polynomial-time algorithm is given for VERTEX COLORING for the class $\mathcal{X}_5$.
- The graphs in $\mathcal{X}_6$ have a simple structure [3,21] which implies easily the existence of a polynomial-time algorithm for VERTEX COLORING that class. Furthermore [14] gives a polynomial-time algorithm for VERTEX COLORING in the larger class $Free(P_5, \text{co-}P_5)$.
- The complexity of VERTEX COLORING in the class $\mathcal{X}_7$ is unknown; it is conjectured in [20] that the problem can be solved in polynomial time in $\mathcal{X}_7$.